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présentée à

L'UNIVERSITE DE PARIS XI
CENTRE D'ORSAY

pour obtenir le grade de

DOCTEUR ES SCIENCES PHYSIQUES

par

Joël SCHERK

Unitarisation et Renormalisation dans le Modèle dual des Résonances

Soutenue le 18 Juin 1971 devant la Commission d'Examen

MM. Ph. MEYER

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C. BOUCHIAT)
R. OMNES)
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TABLE DES MATIERES

I	Introduction	p. 3
II	Unitarisation du Modèle Dual des Résonances	11
III	Renormalisation	23
IV	Références	34
V	Articles Présentés	36
	1 - Zero-Slope Limit of the Dual Resonance Model.	
	2 - Parameter-Free Regularization of One-Loop Unitary Dual Diagram .	
	3 - Renormalization and Unitarity in the Dual Reson- ance Model.	
	4 - Renormalization in the Dual Resonance Model. Its Arbitrariness in the General Case and for Unit Intercept.	
	5 - Mass-Shifts in Second Order of Perturbation Theory in the Dual Resonance Model.	
	6 - Divergence of the Two-Loop Planar Graph in the Dual Resonance Model.	
	7 - Divergence of the N-Loop Planar Graph in the Dual Resonance Model.	

INTRODUCTION

Le concept de dualité connaît depuis quelques années un regain d'intérêt auprès des physiciens théoriciens. Aux débuts de la mécanique quantique, il était question de la dualité onde-corpuscule, et il en est résulté une théorie où un électron peut être considéré soit comme une onde soit comme un corpuscule, mais n'est pas la superposition des deux. Actuellement, dualité signifie qu'une résonance est à la fois résonance dans le canal direct et dans le canal croisé d'une réaction entre particules élémentaires, et par conséquent, que l'amplitude de diffusion ne doit pas être écrite comme une somme de deux termes, mais doit refléter cette propriété de dualité. Il n'est pas à présent encore clair si cette forme nouvelle du concept de dualité est aussi fondamentale que l'ancienne ou constitue seulement un cadre commode pour décrire la dynamique des interactions fortes. Depuis ces trois dernières années, il est remarquable cependant qu'un formalisme mathématique, compliqué certes, mais cohérent, se soit développé en utilisant la dualité comme guide, et se présente actuellement comme une série de perturbations ayant beaucoup de points communs avec celle qu'on

obtient à partir d'une théorie des champs.

Il serait sans doute intéressant d'expliquer comment tous ces développements ont eu lieu à partir de la formule de Veneziano, elle-même venant après des années de phénoménologie à la Regge, l'étude des règles de somme à énergie finie, l'introduction du concept de dualité par Dolen, Horn et Schmid. Néanmoins, il paraît plus aisé de regarder le modèle dual des résonances (M.D.R.) tel qu'il se présente actuellement, indépendamment de son développement historique, car il a déjà acquis une complexité considérable.

Le cadre général dans lequel considérer ce modèle est celui du problème de la construction d'une matrice S satisfaisant les postulats fondamentaux d'invariance de Lorentz, d'unitarité, de symétrie de croisement, d'analyticité et de comportement "à la Regge" à haute énergie. Il paraît actuellement peu probable, tout au moins dans le cadre des mathématiques connues, qu'on puisse réaliser une telle construction au moyen d'expressions fermées, et toutes les tentatives pour réaliser un tel programme se sont fondées sur des approximations. En théorie des champs, on utilise un développement en puissance d'une constante de couplage, et, à chaque étape, on garde analyticité et symétrie de croisement, l'unitarité se trouvant renforcée ordre par ordre. L'hypothèse du comportement à la Regge ne peut être testée qu'en sommant certaines classes de graphes. De plus, en général, la série des perturbations n'est pas convergente, aussi tout énoncé relatif à l'amplitude complète est sujet à caution. Ces difficultés ont conduit un certain

groupe de physiciens à **cesser** de considérer la théorie des champs comme un outil valable pour la construction de la matrice S . On a alors cherché à satisfaire de manière exacte l'unitarité, quitte à perdre la symétrie de croisement et à **renforcer** celle-ci en résolvant de manière **itérative** des équations intégrales. Ces calculs ont mené à l'énoncé d'une hypothèse, le bootstrap, suivant laquelle ce problème de construction de la matrice S a une solution unique, qui décrit précisément le monde réel des particules élémentaires, et où aucune constante n'est laissée arbitraire.

Le modèle **dual des résonances** se situe à mi-chemin entre ces deux approches. Il est similaire à la théorie des champs, car il repose sur un développement perturbatif dans une constante de couplage g qui est laissée arbitraire. En ce sens, il est automatiquement suspect à un partisan du bootstrap, opposé à tout développement de ce type. D'autre part, le terme de Born de ce développement a la propriété essentielle d'avoir le comportement à la Regge ; donc trois des postulats de la matrice S sont satisfaits à chaque étape du calcul et seule l'unitarité qui fait défaut se trouve renforcée à chaque ordre de la théorie des perturbations.

Il est bon à présent, avant de parler de la méthode d'unitarisation, de présenter les propriétés du terme de Born, dont nous partons pour réaliser une telle approche, qui est la formule de Veneziano (1) généralisée à N particules externes. Cette amplitude satisfait la dualité, c'est-à-dire qu'elle peut aussi bien être écrite

comme la somme de contributions de résonances échangées dans la voie s que comme la somme de contributions de résonances échangées dans la voie t . Cette propriété est fondamentale pour la compréhension de la structure mathématique des diagrammes duaux dont nous parlerons plus loin. Ces résonances ont une largeur nulle, donc se présentent dans le terme de Born comme des particules élémentaires, mais la démocratie est assurée entre elles par le fait qu'elles se trouvent toutes sur des trajectoires de Regge linéaires.

$$\alpha(s) = \alpha_0 + \alpha' s - n \qquad n = 0, 1, 2, \dots$$

Trois constantes arbitraires seulement apparaissent dans la théorie : l'intercept α_0 , la pente universelle des trajectoires de Regge α' , la constante de couplage g . La constante fondamentale qui caractérise le fait que nous traitons des interactions fortes n'est pas g mais α' . Le s_0 non spécifié dans l'écriture habituelle du comportement à la Regge $\left(\frac{s}{s_0}\right)^{\alpha(t)}$ est fixé dans le modèle comme $s_0 = 1/\alpha'$. Du point de vue de ce modèle, l'interaction forte est "forte" non à cause d'une grande constante de couplage g , mais à cause de l'existence d'une pente $\alpha' \neq 0$ pour les trajectoires de Regge.

Il est d'ailleurs amusant, mais non surprenant, de constater que si l'on prend la limite où $\alpha' \rightarrow 0$, gardant $g/\sqrt{\alpha'} = \lambda$ fixé, le terme de Born du développement en perturbations, tend vers celui de la théorie des champs en $\lambda\phi^3$. Cette correspondance entre la théorie en ϕ^3 et le M.D.R. servira d'ailleurs de point d'appui utile pour

l'unitarisation (2).

Outre les propriétés de symétrie de croisement et de comportement à la Regge, le M.D.R. possède aussi des propriétés raisonnables à haute énergie en ce qui concerne les réactions inclusives, introduisant en particulier une coupure sur le moment de transfert(3). Cette propriété n'est pas spécifique au M.D.R. et est obtenue dans une large classe d'autres modèles.

Une propriété fondamentale du modèle est la factorisation, c'est-à-dire que les amplitudes à N points qui généralisent l'amplitude de Veneziano peuvent être saturées par un ensemble de résonances indépendant du nombre de pattes externes. Il est remarquable que cette propriété soit satisfaite par le modèle, car elle n'avait pas été postulée a priori. Dans l'unitarisation, cette propriété joue un rôle fondamental, et il y a beaucoup de modèles qui ont généralisé le M.D.R. en respectant les propriétés précédentes mais ont échoué à satisfaire la factorisation. Quand on étudie le spectre on trouve que la dégénérescence du niveau d'ordre N croît exponentiellement en fonction de $\sqrt{N}$, ce qui est un trait commun avec le modèle statistique de Hagedorn. Cette prédiction sur la dégénérescence du spectre des résonances peut avoir d'éventuelles applications cosmologiques, car elle entraîne l'existence d'une température limite $T_0 \approx 180$ MeV (4).

Le spectre du modèle contient aussi des fantômes, c'est-à-dire des particules à métrique négative. Ces particules fantômes

se trouvent présentes de manière intrinsèque, car la factorisation de l'amplitude s'effectue au moyen d'oscillateurs harmoniques a_n^μ , où $\mu = 0, \dots, 3$ et $n = 1, 2, \dots$ satisfaisant les règles de commutation $[a_n^\mu, a_m^{\nu\dagger}] = -g^{\mu\nu} \delta_{mn}$, et les oscillateurs a_m^{+0} créent des particules à métrique négative. Cependant, ce problème n'est pas désespéré, étant commun à bien des théories covariantes, et il est résolu dans le M.D.R. d'une manière similaire à l'électrodynamique, au moyen d'identités de jauge. Il existe en effet dans le modèle trois opérateurs L_0, L_1, L_{-1} qui satisfont les relations de commutation des générateurs du groupe $SU(1,1)$. L'opérateur $L_0 = \sum_{n=1}^{\infty} n a_m^{+\mu} a_n^\mu - \alpha' p^2$ correspond à l'hamiltonien du système, et les états sur la couche de masse, c'est-à-dire les résonances, sont définis par l'équation

$$(L_0 - \alpha_0) | \psi \rangle = n | \psi \rangle \quad n = 0, 1, 2, \dots$$

Cependant ils satisfont aussi l'équation subsidiaire

$$L_1 | \psi \rangle = 0.$$

Cette condition supplémentaire est analogue à l'équation $p^\mu a_\mu | \psi \rangle = 0$ de l'électrodynamique quantique, déterminant la fonction d'onde du photon. On peut en effet montrer qu'il existe dans l'espace engendré par les oscillateurs harmoniques un **espace d'états parasites**, créés par l'opérateur $L_0 - L_{-1}$, c'est-à-dire des états qui sont découplés des résonances apparaissant dans les graphes en arbre. On est donc conduit à distinguer

deux sortes d'états dans cet espace, états "physiques" et états "parasites". On obtient ainsi la condition supplémentaire $L_1 | \psi \rangle = 0$ lorsqu'on définit les états physiques sur la couche de masse. Ce mécanisme ne permet malheureusement d'éliminer que les fantômes créés par l'opérateur a_1^{o+} .

Il existe cependant un cas particulier, dit "cas de Virasoro" (5), où l'espoir existe d'éliminer tous les fantômes. Quand l'intercept est égal à 1, $\alpha_0 = 1$, l'espace des états parasites est beaucoup plus large car il est engendré au moyen d'une infinité d'opérateurs L_n . Les états physiques sur la couche de masse satisfont alors les équations $(L_0 - 1) | \psi \rangle = 0$ et $L_n | \psi \rangle = 0$ $n = 1, 2, 3, \dots$ ce qui restreint beaucoup le nombre des états. En particulier, ces identités entraînent que la particule de masse nulle et de spin 1 du modèle est identique à un photon. Comme il y a une correspondance biunivoque entre les opérateurs a_n^{o+} et L_n , il y a suffisamment d'identités de jauge pour éliminer tous les fantômes. Cependant la conjecture suivant laquelle tous les fantômes sont éliminés n'a pas encore été prouvée, même si elle paraît très plausible ayant été vérifiée jusqu'à la neuvième fille. Il est intéressant de demander que l'invariance par rapport à un groupe de jauge le plus large possible détermine l'intercept $\alpha_0 = 1$, ne laissant que g et α' comme constantes arbitraires. Comme on peut considérer α' comme une constante d'échelle de masse, seul g reste indéterminé.

Cependant, même si les fantômes sont éliminés dans le cas $\alpha_0 = 1$, le terme de Born exhibe un tachyon à $\alpha' m^2 = -1$. On a

donc à choisir dans la meilleure des hypothèses entre une infinité de fantômes et un tachyon. IL semblerait que l'unitarisation demande de choisir la seconde, et permette éventuellement la solution du problème du tachyon ; nous y reviendrons plus loin.

Enfin, avant de présenter l'unitarisation, il faut remarquer que ce H.D.R. du type ϕ^3 n'est pas le plus général, et qu'éventuellement dans d'autres modèles certaines difficultés du modèle du type ϕ^3 peuvent être résolues, par exemple l'absence de tachyon sur une trajectoire à intercept positif dans le modèle de Neveu et Schwarz (6

II UNITARISATION DU MODELE DUAL DES RESONANCES

Pourquoi procéder à l'unitarisation de cette théorie en ϕ^3 ? On peut considérer ce modèle comme l'analogue d'un laboratoire pour théoriciens, de même que les graphes de Feynman sont un laboratoire d'étude de la matrice S en ce qui concerne les propriétés d'analyticité. Ici, nous avons en outre mis comme "input" la dualité et le comportement à la Regge. Des traits qui devraient apparaître dans une théorie à la Feynman seulement en sommant toute la théorie des perturbations, par exemple le Pomeron, peuvent apparaître dans ce modèle dans les graphes à une boucle. Aussi le caractère "consistant" ou non de la théorie doit-il apparaître assez tôt. Par conséquent si le bootstrap est une hypothèse exacte, on peut se trouver dans les situations suivantes :

- aucun modèle dual ne peut être unitarisé de manière conséquente en fonction d'une constante de couplage arbitraire. C'est le point de vue de Chew (7) et aussi de Veneziano (8). L'argument de Veneziano est le suivant : si g est très petit, les largeurs des résonances sont très petites, donc sur l'axe réel on n'a pas à haute énergie le comportement

à la Regge, mais une succession de pics si g est trop petit. Donc g ne peut avoir n'importe quelle valeur et est déterminé par l'unitarité et le comportement à la Regge sur l'axe réel. On doit alors se contenter d'une unitarisation approchée, mais où g peut être déterminé par une condition de bootstrap. Nous reviendrons sur cette hypothèse, et expliquerons pourquoi nous y sommes opposés. Cette attitude peut être décrite comme "bootstrapiste-venezianiste".

- seul un, ou une classe restreinte de modèles, peuvent être unitarisés de manière conséquente, quel que soit g . La condition de conséquence est que la singularité du Pomeron apparaissant dans les graphes non plans soit conciliable avec l'unitarité. Cette condition est très forte et est une condition sur le modèle lui-même. Seul un modèle réaliste, introduisant tous les nombres quantiques observés, et incluant les fermions pourrait alors probablement être unitarisé. C'est une sorte de semi-bootstrapisme venezianiste. Nous reviendrons sur cette conjecture.

- enfin, aucun modèle dual ne peut être unitarisé de manière conséquente, pour diverses raisons, et la solution du problème du bootstrap doit être cherchée dans une autre direction (bootstrap anti-venezianiste).

Nous expliquerons pourquoi nous pensons que la seconde hypothèse peut être raisonnablement soutenue ; le modèle peut alors être considéré comme une sorte de tentative d'expérience dans le bootstrap, en décrivant au moyen d'expressions mathématiques comment les postulats de la matrice S pourraient être réalisés.

Décrivons à présent la méthode d'unitarisation.

L'idée de Kikkawa, Sakita, Virasoro a été de procéder comme en théorie des champs pour $\lambda \phi^3$ en utilisant des "Feynman-like diagrams" (F.L.D.) où une ligne interne représente une "tour de résonances". Ces diagrammes doivent avoir les propriétés de symétrie de croisement, dualité, comportement à la Regge, et satisfaire l'unitarité de manière perturbative.

Un premier problème a été la classification de ces graphes. Ce problème a été résolu par Kikkawa, Klein, Sakita et Virasoro (9 - 10). La zoologie des graphes duaux s'est avérée remarquablement simple et identique à la classification des surfaces compactes à deux dimensions avec frontière. Tous ces graphes peuvent aussi être construits en insérant sur un arbre trois tadpoles et un opérateur de "self-énergie"(11). La classification des graphes duaux est donc simple, et les diagrammes appartiennent à trois catégories : plans non plans orientables, non plans non orientables.

Le comptage des graphes duaux est très différent de celui de la théorie de Feynman, en particulier, il n'y a pas de Lagrangien comme guide pour assurer que l'on puisse construire un développement satisfaisant à la fois dualité et symétrie de croisement. Néanmoins, il a été montré qu'en ce qui concerne l'unitarité sur la singularité dominante, elle peut être assurée conjointement avec la symétrie de croisement, en comptant chaque graphe dual avec le même poids (12). Nous avons montré également que si l'on choisit de compter

chaque graphe dual avec le même poids, alors dans la limite où $\alpha' \rightarrow 0$, $g/\sqrt{\alpha'} = \lambda$ fixé, on retrouve à chaque ordre le comptage de Feynman et l'amplitude de la théorie en ϕ^3 (2). La classification des graphes duaux est donc plus compacte que celle des graphes de Feynman, mais en fait chaque "coin" de la région d'intégration d'un graphe dual correspond à un graphe de Feynman, avec le bon coefficient. Cette correspondance avec la théorie en ϕ^3 fixe donc de manière unique le poids de chaque graphe dans le développement en série de perturbations.

La méthode de construction de chacun de ces graphes repose sur la factorisation. On construit dans l'espace des oscillateurs harmoniques des opérateurs vertex $V(p)$ (deux reggeons- une particule scalaire), propagateurs $D(p)$, twists, et au moyen de ces opérateurs à partir desquels on reproduit les diagrammes en arbre, on obtient les graphes à une boucle.

Un problème longuement débattu est de savoir si dans les états intermédiaires qui apparaissent dans les diagrammes avec boucles, les états que nous avons qualifiés de parasites doivent ou non être inclus. Ces états étant par définition découplés des diagrammes en arbre, on est en principe libre de les inclure ou non dans les boucles puisqu'ils peuvent être éliminés en introduisant d'abord un opérateur de projection (13) sur les états physiques, et au moyen de cet opérateur, de construire un formalisme avec des vertex et des propagateurs ne contenant pas les états parasites, permettant de calculer les diagrammes à N boucles. Ce programme a été mené à bien pour N boucles

dans le cas $\alpha_0 \neq 1$ et pour une boucle dans le cas de Virasoro ($\alpha_0 = 1$) (14). Ce formalisme au moyen d'états projetés respecte bien la dualité pour les graphes en arbre et à une boucle ; néanmoins, on n'est pas assuré a priori qu'il en de même pour les graphes à N boucles, car les opérateurs projetés (vertex à N reggeons) satisfont la dualité de manière globale, c'est-à-dire une fois intégration faite sur les variables qui les définissent. Or à chaque ordre, on doit changer la région d'intégration, et a priori il n'est pas évident que les propriétés de dualité soient maintenues. Cependant Cremmer (15) a montré explicitement que certaines de ces propriétés de dualité locale sont maintenues pour les graphes à deux boucles. Une autre approche consiste à utiliser des vertex à N reggeons qui satisfont de manière locale la dualité, mais incluent des contributions des états parasites (16). La dualité est alors garantie pour les diagrammes à N boucles. Il se peut que les deux approches soient valables, ou une seule d'entre elles, l'important étant d'avoir des amplitudes duales et factorisables. Dans le cas de Virasoro il serait évidemment intéressant d'avoir un formalisme projeté, car si les états projetés sont bien tous des fantômes, il est utile de les éliminer, afin qu'ils ne réapparaissent pas comme pôles dans les diagrammes à deux boucles, par exemple dans un propagateur inséré entre deux boucles. Néanmoins, en ce qui concerne la suite, en particulier les questions concernant le Pomeron et la renormalisation, l'inclusion ou l'élimination des états parasites provoque une modification mineure de l'intégrant des diagrammes à N boucles qui ne modifie pas la discussion de ces questions. Dans la plupart des questions abordées ce problème interviendra donc de manière purement technique.

Passons maintenant à la construction explicite des diagrammes à une boucle qui se trouve essentiellement exposée dans le deuxième article présenté (Renormalization and Unitarity in the D.R.M.). A chaque propagateur est associée une variable x_i comprise entre zéro et un. L'expression de $F_1^{(N)}$ se décompose en deux parties : une partie qui comporte une intégration sur d^4k semblable à celle qu'on effectue sur les graphes de Feynman dans la paramétrisation de Bogoliubov ; une partie qui est la trace du produit des opérateurs vertex, propagateurs, twist, éventuellement opérateur de projection, qui définissent le graphe. L'unitarité perturbative du graphe est assurée par sa construction à la Feynman de sorte que, si l'on coupe une ligne, on retrouve un diagramme en arbre. Le calcul de la trace nécessite des manipulations algébriques reliées à l'algèbre sous-jacente du groupe $SU(1,1)$, et s'effectue de manière commode grâce à l'emploi des états cohérents (17). L'intégration sur d^4k s'effectue au moyen d'une rotation de Wick dont la validité peut être vérifiée a posteriori en étudiant les propriétés analytiques obtenues. Rassemblant ces deux parties on peut écrire l'expression d'un graphe à une boucle sous la forme :

$$F_1^{(N)} = (g_s/\alpha')^N \int_0^1 \prod_{k=1}^N dx_k \mu(x_k, \alpha_0) \exp -\frac{\alpha'}{2} \sum_{i \neq j} p_i \cdot p_j N_8(x_k)$$

L'intégrant comporte deux parties : une partie indépendante des impulsions externes (mesure d'intégration), et une partie qui dépend de ces impulsions.

Décrivons d'abord la mesure d'intégration ; sa particularité essentielle est de faire apparaître la fonction de partition $f(W) = \prod_{n=1}^{\infty} \frac{1}{1 - W^n}$ élevée à la puissance 4. La variable $W = \prod_{i=1}^N x_i$ varie entre 0 et 1. La puissance 4 est caractéristique du nombre d'oscillateurs (quatre dimensions d'espace-temps). Elle se réduit à 3 dans le cas $\alpha_0 = 1$ si les états parasites sont projetés. Dans d'autres modèles elle peut être plus élevée ; la fonction de partition explose quand $w \rightarrow 1$ comme une exponentielle. En fait, cette fonction de partition mathématique est aussi une fonction de partition au sens de la mécanique statistique car

$$f(w) = \text{Tr} \left[e^{-\frac{H}{kT}} \right] \quad \text{où } H = L_0 + \alpha' p^2.$$

La relation entre T et w est $kT = -1/\ln W$; il s'agit là d'une "température" fictive, car H est sans dimension, et décrit une distribution de carrés de masses et non de masses. Néanmoins, l'analogie est commode pour comprendre ce qui se passe. La divergence exponentielle se produit quand la "température" du graphe tend vers l'infini, donc quand on excite un très grand nombre d'états à très haute énergie dans le moment k^2 d'intégration. La limite $T = 0 \rightarrow (w = 0) \rightarrow$ a elle aussi une interprétation, car c'est en développant au voisinage de $w = 0$ l'intégrant, que l'on trouve les seuils d'unitarité. En fait tout le contenu de l'unitarité se trouve concentré au voisinage de $T = 0$, alors que $T \rightarrow \infty$ correspond au comportement à très haute énergie du graphe dans les lignes internes. on ne peut pas considérer cette singularité apparaissant à $T \rightarrow \infty$ comme parasite car le M.D.R. est précisément conçu pour interpoler hautes et basses énergies, et dans son cadre on doit

prendre au sérieux cette singularité. En particulier, on ne peut limiter la région d'intégration sur T , ($T \leq T_0$), ce qui serait compatible avec l'unitarité, mais détruirait le comportement à la Regge à $s \rightarrow +\infty$

Fort heureusement cette fonction a été beaucoup étudiée, et en mécanique statistique, dans le cadre du modèle d'Ising, on a utilisé une propriété essentielle de cette fonction, sous le nom de principe de dualité de Kramers-Vannier. Il s'agit en fait de l'application d'une formule de Hardy-Ramanujan :

$$\sqrt{\frac{2\pi}{|\ln w|}} f(w) = w^{\frac{1}{24}} q^{-\frac{1}{12}} f(q^2) \quad \text{où } q = \exp \frac{2\pi^2}{|\ln w|}$$

la relation entre w et q peut encore s'écrire

$$kT_1 = -\frac{1}{\ln q} = -\frac{1}{2\pi^2} \ln w = \frac{1}{2\pi^2 kT_2}$$

et par conséquent cette identité relie les comportements à haute température et à basse température de cette fonction de partition. Cette égalité permet de trouver le comportement à haute température en fonction du comportement à basse température avec une erreur de l'ordre de q^2 , donc exponentiellement décroissante quand $T \rightarrow \infty$, ($O(\exp - 2\pi^2 kT)$). En mécanique statistique cette propriété de dualité a été étendue à quelques cas, mais il s'agit encore d'une conjecture dans bon nombre de problèmes, pour savoir si une propriété aussi remarquable de la fonction de partition se trouve vérifiée.

Décrivons à présent la partie de l'intégrant

qui dépend des moments externes. La fonction $N_B(x_k)$ dépend en fait de deux variables seulement. Une première variable, $x_{i+1} x_{i+2} \dots x_j = \rho_i / \rho_j$, et la variable w . Elle est la somme de deux morceaux, un qui vient de la trace, l'autre qui provient de l'intégration sur d^4_k . Cherchant un phénomène analogue à celui de la dualité de la fonction de partition, nous avons regardé si deux parties ne se combinaient pas de manière simple. (Voir réfs 18 et 12 où le cas des graphes non plans est examiné). En fait c'est bien le cas, et les deux morceaux se combinent de manière exacte pour donner le logarithme d'une fonction elliptique, la fonction θ de Jacobi (les graphes plans font intervenir θ_1 , les graphes non plans θ_2 et θ_4). L'analogue de la formule de Hardy-Ramanujan est la formule d'inversion des fonctions elliptiques ; après cette inversion N prend une forme encore plus simple, où toute trace d'intégration a disparu, et s'exprime simplement au moyen des variables angulaires $\theta_j - \theta_i = 2\pi \frac{\ln x_{i+1} \dots x_1}{\ln w}$ et la variable q . En particulier on peut obtenir une approximation excellente de cette partie de l'intégrant qui dépend des ρ_i quand $w \rightarrow 1$, ($T \rightarrow \infty$) à l'ordre de q^2 . Avant de discuter les implications de ce résultat, nous allons expliquer comment cette structure mathématique se généralise.

Sous la forme obtenue il est aisé de reconnaître que $N(Z_i, Z_j \mid q)$ n'est autre que la fonction de Neumann d'un disque de rayon unité percé d'un trou circulaire de rayon q et de même centre. Sur ce disque on résout le problème de seconde espèce où le courant (dérivée normale du potentiel) est donné sur la frontière, et la solution

de ce problème est précisément la fonction de Neumann N . On peut également considérer qu'il s'agit d'un problème électrostatique où des charges proportionnelles à p_i , p_j sont placées aux points Z_i , Z_j . Dans le cas des graphes plans, les Z_i sont sur le cercle unité,

$$Z_i = \exp i\theta_i.$$

Dans le cas des graphes non plans orientables, certains des Z_i sont sur le cercle de rayon q ; le cas des graphes non orientables a été étudié par Alessandrini (19).

Nielsen (20) a remarqué que N n'était autre que la fonction de Neumann pour les graphes à zéro boucle, et montré avec Fairlie (21) que les expressions obtenues dans les graphes à une boucle donnaient aussi ces fonctions. Lovelace (22), Alessandrini (19), Amati et Alessandrini (16) ont généralisé ces résultats à N boucles, et montré que la partie de l'intégrant qui dépend des moments externes s'exprime au moyen de la fonction de Neumann d'une surface à deux dimensions. Cette surface est obtenue en traçant le diagramme des quarks du graphe en question, en identifiant les lignes de quarks internes avec les limites de cette surface, et les lignes externes avec des électrodes apportant un courant égal à l'impulsion de la particule incidente. Ce problème est résolu en introduisant le double de la surface (un tore dans le cas du graphe plan à une boucle). On voit ainsi pourquoi les fonctions elliptiques s'introduisent naturellement, et le passage de la variable w à la variable q correspond à couper le

tore suivant deux cycles non équivalents. Un tel changement de variable peut être effectué localement dans le cas des graphes à N boucles pour lesquels les fonctions automorphes généralisent les fonctions elliptiques. Notons également que ces résultats peuvent être retrouvés de manière plus directe dans le formalisme d'intégrales fonctionnelles ; dans ce formalisme on introduit un Lagrangien sur une surface à deux dimensions.

$$\mathcal{L} = -\frac{1}{2} \left[\left(\frac{\partial \tilde{\varphi}}{\partial x} \right)^2 + \left(\frac{\partial \tilde{\varphi}}{\partial y} \right)^2 \right] + i(2\pi)^{1/2} \sum_j p_j \tilde{\varphi}(z) \delta(z - z_j)$$

et l'amplitude est construite au moyen d'une intégrale fonctionnelle sur cette surface (23), du type

$$\iint_D \tilde{\varphi}(x,y) \exp \int dx dy \mathcal{L}(\tilde{\varphi})$$

où D est le domaine d'intégration associé à la surface considérée. Ce formalisme donne les mêmes résultats que le précédent, en particulier pour les diagrammes à une boucle. La fonction de Neumann s'obtient automatiquement, et il est remarquable qu'elle apparait directement sous la forme obtenue après intégration sur 4k , identification et inversion des fonctions θ . Les angles θ_i et la variable q n'ont pas à être obtenus par un changement de variable compliqué, mais apparaissent naturellement. Par contre les propriétés de factorisation, et en particulier le spectre sont plus difficiles à observer dans cette formulation, qui semble donc en quelque sorte duale de celle des opérateurs. Bien que tous ces calculs soient très techniques et souvent compliqués,

il est remarquable que le résultat final, en terme de fonctions harmoniques sur des surfaces à deux dimensions soit simple.

III RENORMALISATION

Revenons à présent au cas des graphes à une boucle.

Au voisinage de $T \rightarrow \infty$, on peut obtenir une approximation de l'intégrant à une erreur qui décroît exponentiellement avec T . Ecrivant cette approximation pour la partie de l'intégrant qui dépend des impulsions externes, on obtient les résultats suivants : Pour les graphes plans et non plans non orientables, la divergence est indépendante des impulsions externes. Ceci n'est pas tout-à-fait surprenant, car ces graphes peuvent être réduits à l'insertion d'un tadpole (plan ou non orientable) sur un diagramme en arbre.

Il n'en est pas de même pour les graphes plans orientables qui peuvent être réduits à un graphe de self-énergie $\Sigma_T(p)$ mais non à un tadpole ; une dépendance sur les impulsions externes s'introduit donc et l'on trouve que pour $\alpha'p^2 \leq -\frac{2}{3}$ le graphe est convergent et peut être défini pour tout p^2 par prolongation analytique (12). Ces deux classes de graphes sont donc très différentes, et

nous allons parler d'abord des graphes plans et non orientables et de leur "renormalisabilité". Nous considérerons le cas des graphes plans pour fixer les idées, mais tout de que nous allons dire s'applique également aux graphes orientables.

La propriété fondamentale dont nous allons nous servir est précisément la possibilité d'obtenir une approximation de $N_B(\rho_j/\rho_1 | w)$ d'ordre q^2 quand $q \rightarrow 0$:

$$N_1(\rho_j/\rho_1; w) = N_1(\theta_j - \theta_1; q) = \tilde{N}(\theta_j - \theta_1) + \mathcal{O}(q^2)$$

Cette séquence d'égalité signifie que la fonction N_1 (1 pour 1 boucle) considérée comme fonction des variables ρ_j/ρ_1 et w peut au moyen de l'inversion des fonctions elliptiques être écrite naturellement comme une fonction des variables $\theta_j - \theta_1$ et q définies précédemment, et peut par conséquent être développée au voisinage de $q = 0$ à l'ordre q^2 .

Définissons alors la partie divergente du graphe de la manière suivante : dans la partie qui dépend des impulsions externes, remplaçons partout N par $\tilde{N}$. On définit ainsi une intégrale $\tilde{F}_1^{(N)}$ qui diverge aussi à $w = 1$. Néanmoins $F_1^{(N)} - \tilde{F}_1^{(N)}$ est fini, car $N_B - \tilde{N} = \mathcal{O}(q^2)$, et la mesure diverge seulement comme $q^{-4/3}$. On peut donc considérer $\tilde{F}_1^{(N)}$ comme la partie divergente du graphe plan à une boucle.

Les propriétés de $\tilde{F}_1^{(N)}$ dépendent de celles de $\tilde{N}$.

Or en fait $\tilde{N}$ est très simple.

$$\tilde{N}(\theta_j - \theta_i) = N_0(\theta_j - \theta_i)$$

N_0 étant la fonction de Neumann du disque sans trou ! Par conséquent la partie de l'intégrant de $\tilde{F}_N$ qui dépend des impulsions externes est identique à celle de la fonction à N points. De cette propriété on déduit que $\tilde{F}_N$ peut s'écrire comme une somme de pôles simples et doubles, et est donc une fonction méromorphe des impulsions externes. Il en résulte que d'une part $\text{Im} \left[F_1^{(N)} - \tilde{F}_1^{(N)} \right] = \text{Im} F_1^{(N)}$, et d'autre part $\tilde{F}_1^{(N)}$ n'introduit pas de fonction entière des variables externes qui détruiraient le comportement à la Regge de $F_1^{(N)}$. De même que $F_1^{(N)}$, $\tilde{F}_1^{(N)}$ est invariant par toute substitution cyclique des impulsions.

Une autre propriété importante de $\tilde{F}_1^{(N)}$ est d'être factorisable. Il s'agit là d'une propriété assez générale qui ne repose pas sur la forme particulière de $\tilde{N}$. Cela signifie que les identités des graphes duaux à une boucle s'applique également à $\tilde{F}_1^{(N)}$, c'est-à-dire qu'il existe un opérateur tadpole et un opérateur de self-energie qui, insérés entre diagramme en arbre, reproduisent $\tilde{F}_1^{(N)}$, comme c'est le cas pour $F_1^{(N)}$ lui-même (24).

Il est remarquable qu'il soit possible de définir de manière naturelle la partie infinie de $F_1^{(N)}$, tout en respectant la dualité, l'unitarité perturbative, la factorisation, et probablement

le comportement à la Regge. Il est assez naturel alors de considérer $\tilde{F}_1^{(N)}$ comme un contreterme (18) et de définir, comme en théorie des perturbations, la partie finie de $F_1^{(N)}$ comme étant $F_1^{(N)} - \tilde{F}_1^{(N)}$. Ce faisant, on ne fait que définir à nouveau la partie réelle de $F_1^{(N)}$ tout en respectant dualité, factorisation et comportement à la Regge. Un tel traitement est également possible pour les graphes non orientables.

Il n'est néanmoins pas possible d'interpréter cette renormalisation de manière simple, car il n'existe pas de Lagrangien à proprement parler à partir duquel on puisse reproduire la série des perturbations. On est donc conduit à se contenter de regarder quelles sont les propriétés de cette renormalisation ordre par ordre.

La première question qui se pose est celle de l'unicité. On peut se convaincre que les contraintes imposées sur $\tilde{F}_1^{(N)}$ laissent en fait pour seul arbitraire la définition de la mesure d'intégration $\tilde{\mu}(x, \alpha_0)$ à une fonction multiplicative près de w :

$$\tilde{\mu} = \mu' g(w).$$

Ceci n'est pas surprenant, car dans le modèle analogue de Fairlie-Nielsen seule la partie dépendant des variables externes est déterminée. Pour déterminer la mesure, il faut imposer en outre que $\tilde{F}_1^{(N)}$ a été construit d'une certaine manière, par exemple au moyen du formalisme d'intégration fonctionnelle. Ceci est possible et implique alors $\mu = \tilde{\mu}$; il n'y a néanmoins pas de raison essentielle pour imposer

cette restriction ; il en résulte que dans le cas général, il existe un degré d'arbitraire infini dans la définition de $F_1^{(N)} - \tilde{F}_1^{(N)}$, et on doit donc considérer la théorie comme non renormalisable. Il existe cependant un cas où le degré d'arbitraire se réduit à une constante (25). Nous avons vu en effet que $\tilde{N}$ ne dépend que des θ_i . Il semblerait donc qu'on puisse séparer les intégrations sur θ_i et w (ou q). En fait ce n'est pas possible à cause de la présence du

$$\text{facteur } \left[\prod_i \frac{1 - x_i}{\sin \frac{\theta_i - \theta_{i-1}}{2}} \right]^{\alpha_0 - 1}$$

Dans le cas où $\alpha_0 = 1$ ce facteur se réduit à 1 et les deux intégrations se séparent. On peut alors écrire :

$$\tilde{F}_1^{(N)} = g^2 C_2 \mathcal{F}_1^{(N)}(p_1, \dots, p_N)$$

$\mathcal{F}_1^{(N)}$ est l'intégrale sur les θ_i et contient la dépendance sur les impulsions externes. C_2 est l'intégrale sur la variable w . Si l'on remplace, dans $\tilde{F}_1^{(N)}$, μ par $\mu g(w)$, où g est arbitraire, seul C_2 est affecté. Donc précisément dans le cas où il existe une infinité d'identités de jauge l'arbitraire est minimal pour la définition de $\tilde{F}_1^{(N)}$. Ceci n'est pas tellement surprenant car on sait que dans ce cas, l'intégrant de Veneziano a une invariance plus large que dans le cas ordinaire (invariance conforme au lieu d'invariance projective). Le facteur gênant qui disparaît dans le cas $\alpha_0 = 1$ est précisément celui qui traduit le fait que pour $\alpha_0 \neq 1$, l'intégrant a seulement l'invariance projective.

Le même phénomène se produit pour le graphe non orientable, et sa partie divergente est également proportionnelle à $\mathcal{F}_1^{(N)}(p_1, \dots, p_N)$. Donc seule la somme des constantes C_2 pour le graphe orientable et le non orientable intervient dans la somme des graphes, et on n'a donc qu'une constante arbitraire.

Il est aussi important de savoir comment l'introduction d'un contreterme modifie le terme de Born ; ainsi on pourrait réellement interpréter la renormalisation en développant ordre par ordre ce terme de Born modifié. Il n'a cependant pas été possible de sommer une série de contretermes. Par contre, on peut regarder à l'ordre g^2 le déplacement de masse de chacun des états du spectre (26). Pour cela, on est amené à résoudre l'équation

$$\det \left[L_0 - \alpha' p^2 - \Sigma(p) + \tilde{\Sigma}(p) \right] = 0.$$

A l'ordre g^2 , on peut négliger l'interférence entre multiplets de masse différente, et il suffit de diagonaliser $\Sigma - \tilde{\Sigma}$ à l'intérieur de chaque multiplet de masse donnée. On trouve alors que pour $\alpha_0 = 1$, $\tilde{\Sigma}$ se comporte comme $\tilde{\sigma} I$ où $\tilde{\sigma}$ est une constante indépendante du multiplet, I la matrice identité. Il en résulte qu'à l'ordre g^2 un contreterme fini déplacerait toutes les masses de la même quantité. Le calcul des déplacements de masse à l'ordre g^4 repose sur une définition cohérente d'un état physique hors de la couche de masse qui n'a pas encore été formulée. (Les problèmes reliés à l'extrapolation hors de la couche de masse n'ont pas encore été résolus dans le cadre de ce modèle).

Par conséquent, pour les diagrammes à une boucle, le contreterme a toutes les bonnes propriétés qu'on peut attendre d'une renormalisation dans le cas $\alpha_0 = 1$. Dans ce cas le terme de Born ne contient (probablement) pas de fantômes, mais contient un tachyon. Une idée naturelle est de fixer la constante arbitraire dont on dispose pour obtenir un intercept renormalisé négatif. Ceci est a priori possible, néanmoins la question importante est de savoir si dans ce cas les fantômes réapparaissent ou non, une fois la série des perturbations sommée. Il n'est pas évident qu'il en soit ainsi, car même si $F_1^{(N)}$ était fini, l'intercept renormalisé serait différent de 1, et la même question se poserait indépendamment de toute renormalisation. Cette question est particulièrement difficile, car elle requiert de connaître la forme générale de l'amplitude unitarisée, et de résoudre le problème de l'extrapolation hors de la couche de masse. Une conjecture serait que, s'il était possible de définir un formalisme projeté sur l'espace des états physiques dans le cas où $\alpha_0 = 1$, alors on pourrait se débarrasser du tachyon tout en gardant l'absence de fantôme ; quoi qu'il en soit, on doit reconnaître qu'il s'agit là d'un traitement essentiellement non perturbatif : $(\delta m^2 \simeq 1 \text{ GeV}^2)$.

Avant de considérer le problème de la renormalisation pour N boucles, notons que ce processus de renormalisation a une interprétation simple dans la limite de la pente nulle. En effet, on a vu

que dans la limite où $\alpha' \rightarrow 0$, $\frac{g}{\sqrt{\alpha'}} = \lambda$ et $\frac{\alpha_0}{\alpha'} = -m^2$ étant fixés, on retrouve la théorie en $\lambda\phi^3$. Comme celle-ci est divergente, il est nécessaire d'introduire une coupure non seulement sur la trace ($f(w) \rightarrow f((1-\epsilon)w)$) mais aussi sur k^2 (par exemple du type $\exp -\frac{k^2}{\Lambda^2}$). Alors on a bien une correspondance entre les deux théories régularisées dans cette limite. On peut voir que seul le coin $w = 0$ de la région d'intégration contribue à la limite de pente nulle ; par conséquent divergence exponentielle et contreterme disparaissent ; néanmoins dans un contreterme fini, la mesure $g(w)$ étant arbitraire, elle peut contenir un facteur du type $\frac{1}{\alpha'^{\frac{1}{2}}}$. La contribution d'un tel contreterme à la limite n'est pas nulle, mais se réduit à l'insertion d'un simple tadpole. Par conséquent, dans la limite de pente nulle, la théorie renormalisée se réduit à celle que l'on obtient à partir du Lagrangien

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \bar{\phi})(\partial^\mu \bar{\phi}) + \frac{m^2}{2} \bar{\phi}^2 + \frac{\lambda}{6} \bar{\phi}^3 + c \bar{\phi}$$

qui est précisément le Lagrangien de la théorie en ϕ^3 renormalisée. Le terme $c\bar{\phi}$ correspond à l'introduction du contreterme.

En présence d'un tel terme, m^2 n'est pas la masse physique m_p^2 ; il est amusant de remarquer qu'on obtient deux solutions pour m_p^2 , qui dans la limite $c \rightarrow 0$ tendent vers $+m^2$ et $-m^2$. Il est bien sûr très prématuré d'en conclure que le tachyon peut devenir une particule physique dans le M.D.R. par un mécanisme similaire.

Le premier pas pour étudier la "renormalisabilité" des graphes à N boucles consiste à trouver les divergences de ces diagrammes. Le problème est assez complexe car les "fonctions de partition" généralisées qui contiennent ces divergences peuvent en principe présenter une infinité de divergences exponentielles simultanées. Ce cas correspondrait bien sûr à une théorie non renormalisable. En fait, M. Kaku et moi-même (27) avons démontré que pour $N = 2$ boucles, seulement trois divergences du type précédent apparaissent. Il existe donc dans ce cas une divergence supplémentaire, que Cremmer et Neveu (28) ont renormalisée. Dans le cas de N boucles nous avons montré (29) l'existence de $2^N - 1$ divergences et Alessandrini et Amati ont montré (16) qu'en fait seulement $N + 1$ de ces divergences sont caractéristiques des graphes à N boucles, en ce sens que les autres n'apparaissent que si certains multiplicateurs sont nuls. Il est par conséquent possible de renormaliser la série des perturbations, et j'ai montré que dans le cas $\alpha_0 = 1$ l'introduction d'un contreterme supplémentaire revient à modifier la constante C_2 en $C_2 + g^2 C_4$. Il semble donc que toute la série des perturbations puisse dans le cas $\alpha_0 = 1$ être renormalisée au moyen d'une constante arbitraire (25)

La renormalisabilité de la théorie est certainement liée au comportement de la partie absorptive. Une conjecture, à présent abandonnée, était que cette partie absorptive croissait exponentiellement et que cela expliquait la divergence exponentielle. En fait des calculs explicites sur "computer" par Shapiro et Thorn ont montré que la partie imaginaire n'explose pas. On peut aussi étudier directement la

trajectoire principale ; à l'ordre g^2 , $\alpha(s)$ est remplacé par $\alpha(s) + g^2 \sigma(s)$

Le déplacement $\sigma(s)$ est donné par une intégrale divergente, mais en fait

$\frac{d}{ds} \sigma(s)$ est fini. La renormalisation revient précisément à changer σ en $\sigma(s) - \tilde{\sigma}$, où $\tilde{\sigma}$ est constant. De plus on peut montrer que $\sigma - \tilde{\sigma}$ se comporte comme $(-s)^{1/6}$ dans toute direction du plan complexe sauf sur l'axe réel (18). Si ce comportement est valable aussi sur l'axe réel on en déduit que $\text{Im } \sigma(s) \sim s^{1/6}$, ce qui est compatible avec une relation de dispersion une fois soustraite ; on voit également que la largeur des résonances décroît à haute énergie ($\text{Im } \sigma \propto \sqrt{s} \Gamma(s)$), mais que différents multiplets peuvent interférer puisque $\text{Im } \sigma$ croît. Ceci est en contradiction avec l'hypothèse de Veneziano, et nous tenons pour probable que le comportement à la Regge est réalisé, même pour g très petit, sur l'axe réel.

On peut donc considérer qu'en ce qui concerne les graphes plans et non orientables, dans le cas $\alpha_0 = 1$, les problèmes de l'unitarisation peuvent être surmontés. Il n'en est cependant pas de même des graphes non plans orientables (16). Ceux-ci en effet sont finis mais présentent une nouvelle singularité. Il est tentant d'associer celle-ci au Pomeron ; en effet, elle se produit dans le canal qui aurait les nombres quantiques du vide si on attachait des nombres quantiques aux lignes externes. De plus l'expression de cette coupure de Regge est $\alpha_p(s) = \frac{1}{3} + \frac{\alpha'}{2}s$. L'intercept $1/3$ est indépendant de α_0 mais dépend du nombre d'oscillateurs du modèle ; par contre il est remarquable que $\alpha'_p = \frac{\alpha'}{2}$ soit indépendant du modèle considéré. Cette singularité se présente également comme une coupure dans le plan s, t

et introduit une partie imaginaire qui contredit la relation d'unitarité. Aucune solution n'a été trouvée à ce problème dans la théorie du type ϕ^3 ; il est probable qu'il s'agit là d'une difficulté inhérente à cette théorie et par conséquent on doit conclure que celle-ci ne peut être unitarisée de manière conséquente. Une attitude plus positive consiste à supposer que dans une théorie incluant les nombres quantiques de manière réaliste, ce problème peut être résolu, ce qui est une sorte de bootstrapisme venezianiste ! Cet espoir existe réellement ; ainsi Lovelace (30) a montré que dans des modèles à nombre d'oscillateurs plus élevé, le Pomeron, au lieu d'être une coupure, peut se réduire à un pôle et est alors conséquent avec l'unitarité. A nouveau, la contrainte de factorisation implique $\alpha_0 = 1$. Il se pourrait donc qu'un modèle plus réaliste permette de résoudre ce problème, et les récentes généralisations factorisables du M.D.R. permettent d'avoir quelque espoir dans ce sens.

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ARTICLES PRESENTES

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Parameter-Free Regularization of One-Loop Unitary Dual Diagram*

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We propose a parameter-free regularization of the one-loop planar unitary dual diagram, very similar to the renormalization in quantum field theory, compatible with duality, Regge behavior, unitarity, and crossing symmetry.

I. INTRODUCTION

RECENTLY, there have been several attempts¹⁻³ to put the Veneziano model on the same footing as quantum electrodynamics, by considering the N -point functions as tree diagrams of the theory. This leads to a finite result when factorization problems are neglected.¹ However, when factorization is taken into

account, the one-loop diagram exhibits an exponential divergence due to the large degeneracy of levels.^{2,3}

It would then seem that a renormalization procedure would need an infinite number of subtractions, and thus an infinite number of parameters. However, we show that in the case of the one-loop diagram, the divergent part of the new trajectory, $\alpha_t + g^2 \Sigma(\alpha_t)$, is independent of t and can be removed by a single subtraction. This can be extended to the amplitude itself and one finds that the subtraction of a crossing-symmetric, Regge-behaved, dual amplitude, having only single and double poles in the external variables s and t , is enough to make it finite. We then find that the renormalized amplitude still Reggeizes at $s \rightarrow -\infty$. We have extended this to the case of the one-loop diagram with N external legs.

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¹ K. Kikkawa, B. Sakita, and M. Virasoro, *Phys. Rev.* **184**, 1701 (1969); K. Kikkawa, S. Klein, B. Sakita, and M. Virasoro, University of Wisconsin Report No. 248, 1969 (unpublished).

² D. Amati, C. Bouchiat, and J. L. Gervais, *Nuovo Cimento Letters* **2**, 399 (1969).

³ K. Bardakci, M. B. Halpern, and J. A. Shapiro, *Phys. Rev.* **185**, 1910 (1969).

Our regularization technique is very different from any procedure using finite cutoffs, which essentially amounts to the subtraction of an entire function of s and t .

Because we subtract a sum of poles and double poles, our process is rather similar to the renormalization of second-order diagrams in field theory. So, one may hope that the higher-order diagrams can be made finite by the counter terms thus defined.

II. FORMAL REGGEIZATION OF SECOND-ORDER FOUR-POINT FUNCTION AT $s \rightarrow -\infty$

We start with the second-order four-point function as computed in Refs. 1 and 2:

$$F(p_1, p_2, p_3, p_4) = -ig^4 \int d^4k \int_0^1 dx_1 dx_2 dx_3 dx_4 \\ \times x_1^{-\alpha(k^2)-1} \dots x_4^{-\alpha(k^2)-1} \prod_{i=1}^4 (1-x_i)^{-c} q_0^{-4} (\sqrt{w}) \\ \times \exp\left(-\sum_n \sum_{i,j=1}^4 \frac{C_{ij}^n p_i p_j}{n(1-w^n)}\right), \quad (1)$$

where

$$q_0(Z) = \prod_{n=1}^{\infty} (1-Z^{2n}), \\ w = x_1 x_2 x_3 x_4, \\ C_{ij} = x_1 \dots x_i \quad \text{if } j=4 \\ = (x_1 \dots x_i)(x_{j+1} \dots x_4) \quad \text{if } i \leq j \leq 3 \\ = x_{j+1} \dots x_i \quad \text{if } i > j, \\ \alpha(k^2) = \alpha_0 - \frac{1}{2}k^2.$$

When all the relations found by Fubini and Veneziano are taken into account (Ref. 3), one finds the same expression except for an extra $(1-w)$ factor. This alteration has no consequences for the rest of this paper.

Expression (1) diverges near $w \rightarrow 1$ as $\exp[\frac{2}{3}\pi^2/(1-w)]$. We introduce the temporary cutoff $\theta(4-\epsilon-x_1-x_2-x_3-x_4)$. This cutoff, as already noticed, does not change the imaginary part and preserves Reggeization at $s \rightarrow -\infty$.

After Wick rotation and integration over d^4k , one finds

$$F(\alpha_s, \alpha_t) = 4\pi^2 g^4 \int_0^1 \prod_{i=1}^4 dx_i x_i^{-\alpha_0-1} (1-x_i)^{-c} \frac{q_0^{-4}}{\ln^2 w} e^{\alpha_0 h} T_1^{-\alpha_s} \\ \times T_2^{-\alpha_t} \theta(4-\epsilon-x_1-x_2-x_3-x_4), \quad (2)$$

where

$$h(x_i) = \frac{1}{2} \left(\ln w - \sum_i \frac{\ln^2 x_i}{\ln w} \right) \\ + \sum_n \sum_i \frac{-8w^n + x_i^n + w^n/x_i^n}{n(1-w^n)},$$

$$T_1(x_1, x_2, x_3, x_4) = \exp \frac{\ln x_1 \ln x_3}{\ln w} \\ - \sum_{n=1}^{\infty} \frac{(1-x_1^n)(1-x_3^n)(x_2^n+x_4^n)}{n(1-w^n)}, \quad (3)$$

$$T_2(x_1, x_2, x_3, x_4) = T_1(x_2, x_1, x_4, x_3), \\ \alpha_s = \alpha_0 - \frac{1}{2}(p_1 + p_2)^2.$$

$e^{\alpha_0 h}$, T_1 , and T_2 are well-behaved functions when $w \rightarrow 0$. When $s \rightarrow -\infty$, $F(\alpha_s, \alpha_t)$ Reggeizes and one finds neglecting terms of order $(-\alpha_s)^{\alpha_t-1} \ln s$,

$$F(\alpha_s, \alpha_t) = g^4 (-\alpha_s)^{\alpha_t} [\ln(-\alpha_s) \Gamma(-\alpha_t) \Sigma(\alpha_t) + \Gamma(-\alpha_t) \beta(t) \\ - \Gamma'(-\alpha_t) \Sigma(\alpha_t)] + O(\alpha_s^{\alpha_t-1} \ln s). \quad (4)$$

This is the term in g^4 of the expansion in powers of g^2

$$g^2 \beta_{\text{new}}(t) \Gamma(-\alpha_{\text{new}}) (-\alpha_s)^{\alpha_{\text{new}}(t)},$$

where

$$\alpha_{\text{new}}(t) = \alpha_t + g^2 \Sigma(\alpha_t) + O(g^4)$$

is the new trajectory,

$$\beta_{\text{new}}(t) = 1 + g^2 \beta(t) + O(g^4)$$

is the new residue,

$$\Sigma(\alpha_t) = 4\pi^2 \int_0^1 \frac{dx_2 dx_4}{\ln^2 x_2 x_4} (x_2 x_4)^{-\alpha_0-1} (1-x_2)^{-c} (1-x_4)^{-c} \\ \times e^{\alpha_0 h_1} q_0^{-4} (x_2 x_4)^{1/2} e^{\alpha_t H(x_2, x_4)} \\ \times \theta(2-\epsilon-x_2-x_4), \quad (6)$$

$$h_1 = \frac{\ln x_2 \ln x_4}{\ln x_2 x_4} + 2 \sum_n \frac{-2x_2^n x_4^n + x_2^n + x_4^n}{n(1-x_2^n x_4^n)},$$

and

$$H(x_2, x_4) = 2 \sum_{n=1}^{\infty} \frac{2x_2^n x_4^n - x_2^n - x_4^n}{n(1-x_2^n x_4^n)} - \frac{\ln x_2 \ln x_4}{\ln x_2 x_4} \\ + \ln \left(\sum_{n=1}^{\infty} \frac{n(x_2^n + x_4^n)}{1-x_2^n x_4^n} - \frac{1}{\ln x_2 x_4} \right). \quad (7)$$

$\beta(t)$ has a similar expression.

$\Sigma(\alpha_t)$ is also exponentially divergent near $x_2 x_4 \simeq 1$ and we study the properties of this integral in Sec. III.

III. REGULARIZATION OF $\Sigma(\alpha_t)$

$e^{H(x_2, x_4)}$ is given in terms of T_1 and T_2 by

$$e^{H(x_2, x_4)} = - \frac{\partial^2}{\partial x_1 \partial x_3} T_1 \bigg|_{x_1=x_3=1} \left(\frac{\partial^2}{\partial x_1 \partial x_3} T_2 \bigg|_{x_1=x_3=1} \right)^{-1}. \quad (8)$$

Using the theory of elliptic functions, it is shown in the

Appendix that T_1 can be written

$$T_1 = \frac{\sin(\pi \ln x_2 / \ln w) \sin(\pi \ln x_4 / \ln w)}{\sin(\pi \ln x_1 x_2 / \ln w) \sin(\pi \ln x_1 x_4 / \ln w)} \times \prod_{n=1}^{\infty} \frac{[1 - 2q^{2n} \cos(2\pi \ln x_2 / \ln w) + q^{4n}][1 - 2q^{2n} \cos(2\pi \ln x_4 / \ln w) + q^{4n}]}{[1 - 2q^{2n} \cos(2\pi \ln x_1 x_2 / \ln w) + q^{4n}][1 - 2q^{2n} \cos(2\pi \ln x_1 x_4 / \ln w) + q^{4n}]}, \quad (9)$$

where $q = e^{2\pi^2 / Lnw}$. Notice that the infinite product is equal to 1 up to a null asymptotic expansion in $1-w$. More precisely, it is equal to $1 + O(e^{-2\pi^2 / (1-w)})$ near $w \simeq 1$. By computing $e^{H(x_2, x_4)}$, one finds that the sines drop out and that $e^{H(x_2, x_4)} = 1 + O(e^{-4\pi^2 / (1-x_2 x_4)})$ near $x_2 x_4 \simeq 1$.

Thus, the divergent part of $\Sigma(\alpha_t)$ is independent of t when $\epsilon \rightarrow 0$. Moreover, a single subtraction is enough to make $\Sigma(\alpha_t)$ finite.

$\Sigma(\alpha_t)$ appears as the renormalization of the leading trajectory. Since the scalar external particles belong to it and are stable, we demand that their mass be the physical mass. Thus, we subtract the divergent quantity $\Sigma(0)$.

Since

$$q_0^{-4}(x_2 x_4)^{1/2} [e^{\alpha_t H(x_2, x_4)} - 1] = O\left(\exp\left(-\frac{10}{3} \frac{\pi^2}{1-x_2 x_4}\right)\right) \quad \text{as } x_2 x_4 \rightarrow 1,$$

$\Sigma(\alpha_t) - \Sigma(0)$ is now given by a convergent integral for t below threshold. Above threshold, it can be shown by using standard techniques that $\Sigma(\alpha_t) - \Sigma(0)$ has branch points for every two-particle threshold and pseudo-threshold.⁴

All this regularization supposes that there is only one stable particle in the theory, or equivalently, that the intercept α_0 of the leading trajectory is not too negative. If there were more than one stable particle, we would have to add new parameters in the theory from the beginning, for instance by adding satellites, in order to fit the renormalized values of the masses of stable particles with their physical values.

IV. RENORMALIZATION OF AMPLITUDE $F(\alpha_s, \alpha_t)$

Following the guidelines of the usual Feynman graphs renormalization, we look for a function $\tilde{F}(\alpha_s, \alpha_t)$ obeying the following conditions:

(1) $F - \tilde{F}$ is finite for the cutoff ϵ going to zero.

(2) $\tilde{F}$ is crossing symmetric and in each channel can be written as a sum of single and double poles, the residues at the poles being polynomials in the other variable.

⁴ For $t \rightarrow -\infty$ one can show that $\Sigma(\alpha_t)$ behaves like $(-t)^{1/6}$ up to logarithmic factors. Compare with

$$\int_{-\infty}^{+\infty} e^{bx} (e^{te^{-ax}} - 1) dx = (1/a) \Gamma(-b/a) (-t)^{b/a} \quad \text{for } a > b > 0.$$

(3) $F - \tilde{F}$ Reggeizes at $s \rightarrow -\infty$.

(4) $\tilde{\Sigma}(\alpha_t) = \Sigma(0)$ so that $F - \tilde{F}$ behaves like $g^4(-\alpha_s)^{\alpha_t} \times \ln(-\alpha_s) \Gamma(-\alpha_t) [\Sigma(\alpha_t) - \Sigma(0)]$ when $s \rightarrow -\infty$.

Notice that in expression (9) the poles and the double poles in the s channel arise from the behavior of the sines when $x_2 = 1$ or $x_4 = 1$, whereas the infinite product gives rise to the cut when $w \rightarrow 0$.

We are thus led to define a function $\tilde{F}(\alpha_s, \alpha_t)$ by Eq. (2) except that T_1 and T_2 are replaced by

$$\tilde{T}_1 = \frac{\sin(\pi \ln x_2 / \ln w) \sin(\pi \ln x_4 / \ln w)}{\sin(\pi \ln x_1 x_2 / \ln w) \sin(\pi \ln x_1 x_4 / \ln w)}, \quad (10)$$

$$\tilde{T}_2 = \tilde{T}_1(x_2, x_1, x_4, x_3).$$

$F - \tilde{F}$ is evidently finite when $\epsilon \rightarrow 0$, because near $w = 1$

$$\tilde{T}_1 - T_1 = O(e^{4\pi^2 / Lnw}).$$

$\tilde{F}$ has poles and double poles located like those of F . $\tilde{T}_1$ and $\tilde{T}_2$ have the following property:

$$\tilde{T}_2 = 1 - \tilde{T}_1.$$

From the set of variables x_i , we change to the set x_1, x_3, z_2 , and z_4 , defined by⁵

$$\begin{aligned} z_2 &= \frac{\sin(\frac{1}{2}\pi \ln x_2 / \ln w) \sin(\frac{1}{2}\pi \ln x_1 x_2 x_3 / \ln w)}{\sin(\frac{1}{2}\pi \ln x_2 x_1 / \ln w) \sin(\frac{1}{2}\pi \ln x_2 x_3 / \ln w)}, \\ z_4 &= \frac{\sin(\frac{1}{2}\pi \ln x_4 / \ln w) \sin(\frac{1}{2}\pi \ln x_3 x_4 x_1 / \ln w)}{\sin(\frac{1}{2}\pi \ln x_1 x_4 / \ln w) \sin(\frac{1}{2}\pi \ln x_3 x_4 / \ln w)}, \end{aligned} \quad (11)$$

so that

$$0 \leq z_2 \leq 1, \quad 0 \leq z_4 \leq 1$$

and

$$\tilde{T}_1 = z_4 z_2, \quad \tilde{T}_2 = 1 - z_2 z_4.$$

By expanding in powers of z_2 and z_4 , and integrating over z_2 and z_4 , we find that $\tilde{F}$ is of the form

$$\begin{aligned} \tilde{F}(\alpha_s, \alpha_t) &= g^4 \sum_{n, p, q \geq 0} \frac{1}{n+p-\alpha_s} \frac{1}{n+q-\alpha_s} \frac{\Gamma(\alpha_t + n + 1)}{n! \Gamma(\alpha_t + 1)} \\ &\quad \times \int_0^1 dx_2 dx_4 \tilde{J}_{pq}(x_2, x_4). \end{aligned} \quad (12)$$

This shows that $\tilde{F}$ has the very structure of a counter-

⁵ This change is somewhat similar to the one defined by Eq. (3.7) in Ref. 1.

term in the usual renormalization theory. As for F itself, the same change of variables casts it into the form

$$F(\alpha_s, a_t) = g^4 \sum_{n, p, q \geq 0} \frac{1}{n+p-\alpha_s} \frac{1}{n+q-\alpha_s} \times P_n(\alpha_s, p, q | \alpha_t), \quad (13)$$

where $P_n(\alpha_s, p, q | \alpha_t)$ is a polynomial of degree n in α_t , whose coefficients are functions of p , q , and α_s , and have unitarity cuts in the α_s plane.

One can check that $\tilde{F}$ Reggeizes for $s \rightarrow -\infty$,⁶ and that the equality $\tilde{\Sigma}(\alpha_t) = \Sigma(0)$ is verified.

Notice that $\tilde{F}$ is not uniquely determined by conditions (1)–(4). One can always add to F a finite dual function, behaving like $(-\alpha_s)^{\alpha_t-1}$. Although we could not prove it, it seems very likely that it amounts to a change on the Born term. One encounters a similar problem in Lagrangian field theory for the definition of counterterms.

V. REGULARIZATION OF N -POINT FUNCTION

We integrate first the expression of $F_1^{(N)}(p_1, \dots, p_N)$ over d^4k . We express then the $p_i p_j$ in terms of the variables⁷

$$\alpha_{ij} = \alpha_0 - \frac{1}{2}(p_i + \dots + p_j)^2, \quad i < j.$$

Then

$$F_1^{(N)}(p_1 \dots p_N) = 4\pi^2 g^N \int_0^1 \prod_{i=1}^N dx_i \prod_{i,j; i < j} T_{ij}^{-\alpha_{ij}} \times F_N(x_i) \theta(N - \epsilon - \sum x_i), \quad (14)$$

where $F_N(x_i)$ is independent of the α_{ij} .

We call T_{ij} the conjugate variable to the trajectory α_{ij} . One finds (see the Appendix) that

$$T_{ij} = \theta_1 \left(\frac{\nu_{ij}}{\tau} \middle| -\frac{1}{\tau} \right) \theta_1 \left(\frac{\nu_{i-1, j+1}}{\tau} \middle| -\frac{1}{\tau} \right) / \theta_1 \left(\frac{\nu_{i, j+1}}{\tau} \middle| -\frac{1}{\tau} \right) \theta_1 \left(\frac{\nu_{i-1, j}}{\tau} \middle| -\frac{1}{\tau} \right). \quad (15)$$

Keeping only the sines in the expansion of θ_1 , we define $\tilde{F}_1^{(N)}(p_1 \dots p_N)$ by (14) with T_{ij} replaced by

$$\tilde{T}_{ij} = \frac{\sin(\pi \ln x_{i+1} \dots x_j / \ln w) \sin(\pi \ln x_i \dots x_{j+1} / \ln w)}{\sin(\pi \ln x_{i+1} \dots x_{j+1} / \ln w) \sin(\pi \ln x_i \dots x_j / \ln w)}. \quad (16)$$

$F_1^{(N)} - \tilde{F}_1^{(N)}$ is evidently finite for $\epsilon \rightarrow 0$. Let us look now at the factorization properties of $\tilde{F}_1^{(N)}$.

⁶ Whether the renormalized amplitude Reggeizes in any direction of the complex s plane except the positive real axis is still an open question, because of the singularities of θ functions one could encounter when distorting the integration contour over x_1 and x_3 .
⁷ Chan Hong-Mo and Tsou Sheung Tsun, Phys. Letters **28B**, 485 (1969).

We have seen in the case $N=4$ that $\tilde{T}_2 = 1 - \tilde{T}_1$; this can be generalized, and the $\tilde{T}_{ij}$ satisfy the general equation of duality:

$$T_{ij} = 1 - \prod_{p,q} T_{p,q} \prod_{r,s} T_{r,s}, \quad 1 \leq p < i, \quad i \leq q < j, \quad i < r \leq j, \quad j < s \leq N-1. \quad (17)$$

This can be shown from the fact that

$$1 - T_{ij} = \frac{\sin(\pi \ln x_i / \ln w) \sin(\pi \ln x_{j+1} / \ln w)}{\sin(\pi \ln x_{i+1} \dots x_{j+1} / \ln w) \sin(\pi \ln x_i \dots x_j / \ln w)}$$

and that

$$T_{p,q} T_{p,q+1} \dots T_{p,q+r} = \frac{\sin(\pi \ln x_{p+1} \dots x_q / \ln w) \sin(\pi \ln x_p \dots x_{q+r+1} / \ln w)}{\sin(\pi \ln x_p \dots x_q / \ln w) \sin(\pi \ln x_{p+1} \dots x_{q+r+1} / \ln w)}.$$

The T_{ij} can be expressed in terms of the $N-3$ T_{ij} ($2 \leq j \leq N-2$) by the usual formulas. To see the poles and double poles in the $\alpha_{1,p}$ channel, we write, as previously, $T_{1,p} = z_1 z_2$ and we choose two other independent variables ξ_1, ξ_2 .

After development in powers of z_1^r, z_1^q and integration over z_1, z_2, ξ_1, ξ_2 , we get

$$F_1^{(N)} = \sum_{\{m\}} \sum_{r,s=0}^{+\infty} \int_0^1 \prod_{i=1; i \neq p}^{N-2} dT_{1,i} T_{1,i}^{-\alpha_{1,i}-1} (1 - T_{1,i})^{-\epsilon} \times \langle 0 | V(p_{N-1}) T_{1,N-2}^H \dots V(p_{p+1}) | \{m\} \rangle \times \frac{f_{r,s}^{(p)}(T_{1,i})}{(M+r-\alpha_{1p})(M+s-\alpha_{1s})} \times \langle \{m\} | V(p_{p+1}) T_{1,p-1}^H \dots V(p_2) | 0 \rangle. \quad (18)$$

The sum is taken over the occupation number basis, and M is defined by $H | \{m\} \rangle = M | \{m\} \rangle$.

The residue $f_{r,s}^{(p)}$ is independent of the external variables $\alpha_{i,j}$.

VI. CONCLUSION

We have shown that, by a procedure very similar to the renormalization in quantum field theory, the N -point one-loop amplitude can be made convergent. The renormalized amplitude satisfies perturbative unitarity, duality, and Regge behavior. The next step is to study the possibility of the renormalization of the nonplanar diagrams and of the N -loop diagrams. This problem is under investigation.

APPENDIX: COMPUTATION OF T_{ij}

We compute here the conjugate variable to the trajectory

$$\alpha_{i,j} = \alpha_0 - \frac{1}{2}(p_i + \dots + p_j)^2.$$

The notations are those of Tannery and Molk (Ref. 8). The identities

$$p_i p_{i+j} = -\alpha_{i,i+j} + \alpha_{i,i+j-1} + \alpha_{i+1,i+j} - \alpha_{i+1,i+j-1} \quad (j > 2), \quad (19)$$

$$p_i p_{i+2} = -\alpha_{i,i+2} + \alpha_{i,i+1} + \alpha_{i+1,i+2}, \quad (20)$$

$$p_i p_{i+1} = -\alpha_{i,i+1} - \alpha_0 \quad (21)$$

show that the integration over d^4k gives

$$\int d^4k x_1^{-\alpha(k^2)-1} x_N^{-\alpha(k+p_{N-1})^2-1} = \exp\left(-\sum_{i,j} \alpha_{i,j} \frac{\ln x_i \ln x_{j+1}}{\ln w}\right) f(x_i),$$

where f depends only of x_i and α_0 .

We now compute

$$I = \exp\left[-\sum_{i,j} C_{ij} \frac{p^i p_j}{n(1-w^n)}\right].$$

We note first that for $i < j$, $C_{ji} C_{ij} = w$, so that

$$A_{ij} = \prod_n \exp\left(-\frac{C_{ij}^n + C_{ji}^n}{n(1-w^n)}\right) = (1-x_{i+1} \cdots x_j) \times \prod_{n=1}^{\infty} \left(1 - \frac{w^n}{x_{i+1} \cdots x_j}\right) (1-w^n x_{i+1} \cdots x_j).$$

From Vol. II, p. 250, xxix (1) in Ref. 8,

$$A_{ij} = -2i\pi Z_{ij} q_0^2 \frac{\theta_1(\nu_{ij}|\tau)}{\theta_1'(0|\tau)}, \quad (22)$$

where

$$q_0 = \prod (1-q^{2n}), \quad q = \sqrt{w}, \quad Z_{ij} = (x_{i+1} \cdots x_j)^{1/2}, \\ \nu_{ij} = (1/2i\pi) \ln x_{i+1} \cdots x_j, \quad \tau = (1/2i\pi) \ln w, \\ \theta_1(\nu) = 2q_0 q^{1/4} \sin \nu \pi \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\nu \pi + q^{4n}). \quad (23)$$

⁸ J. Tannery and J. Molk, *Elements de la Théorie des Fonctions Elliptiques* (Gauthier-Villars, Paris, 1893).

On the other hand,

$$A_{ii} = \exp\left(-\sum_n \frac{w^n}{n(1-w^n)}\right) = q_0,$$

so that

$$I = \prod_{i < j} \left[-2i\pi Z_{ij} \frac{\theta_1(\nu_{ij}|\tau)}{\theta_1'(0|\tau)} \right]^{p_i p_j},$$

where q_0 has dropped out owing to momentum conservation. Applying the identities (19)–(21), one finds that this can be rewritten

$$I = \prod_{i < j} \left[\frac{\theta_1(\nu_{ij}|\tau) \theta_1(\nu_{i-1,j+1}|\tau)}{\theta_1(\nu_{i,j+1}|\tau) \theta_1(\nu_{i-1,j}|\tau)} \frac{Z_{ij} Z_{i-1,j+1}}{Z_{i,j+1} Z_{i-1,j}} \right]^{\alpha_{i,j}} g(x_i),$$

where g depends only on the x_i and on α_0 .

We apply now the transformation formula obtained by the theory of the elliptic functions:

$$\theta_1(\nu|\tau) = i \left(\frac{i}{\tau}\right)^{1/2} e^{-i\pi(\nu^2/\tau)} \theta_1\left(\frac{\nu}{\tau} \middle| -\frac{1}{\tau}\right).$$

One finds then that the conjugate factor to $\alpha_{i,j}$ is

$$\exp\left(-\frac{\ln x_i \ln x_{j+1}}{\ln w}\right) \left[\theta_1\left(\frac{\nu_{ij}}{\tau} \middle| -\frac{1}{\tau}\right) \theta_1\left(\frac{\nu_{i-1,j+1}}{\tau} \middle| -\frac{1}{\tau}\right) / \theta_1\left(\frac{\nu_{i,j+1}}{\tau} \middle| -\frac{1}{\tau}\right) \theta_1\left(\frac{\nu_{i-1,j}}{\tau} \middle| -\frac{1}{\tau}\right) \right]. \quad (24)$$

Collecting the term coming from the integration over d^4k , we find that the conjugate variable to the trajectory α_{ij} is

$$\theta_1\left(\frac{\nu_{ij}}{\tau} \middle| -\frac{1}{\tau}\right) \theta_1\left(\frac{\nu_{i-1,j+1}}{\tau} \middle| -\frac{1}{\tau}\right) / \theta_1\left(\frac{\nu_{i,j+1}}{\tau} \middle| -\frac{1}{\tau}\right) \theta_1\left(\frac{\nu_{i-1,j}}{\tau} \middle| -\frac{1}{\tau}\right). \quad (25)$$

Using (23), we then prove (9).

Renormalization and Unitarity in the Dual-Resonance Model*

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All the one-loop graphs of the dual-resonance model are explicitly calculated. These graphs fall into three categories: planar, nonorientable, and orientable nonplanar. Using the properties of various elliptic functions, we are able to generalize the renormalization procedure, obtained previously for the planar diagrams, to the other two categories. The orientable nonplanar diagrams turn out to be particularly interesting. First, their integration regions have to be reduced from the ones naively obtained in order to avoid multiple counting. Secondly, they give rise to new singularities (branch points) in channels that are naturally identified as having vacuum quantum numbers. These singularities are probably related to the Pomeranchukon. The question of unitarity is explored at the one-loop level, i.e., to the first nontrivial order in the perturbation series. Although the counting of diagrams is somewhat subtle, a rather simple result emerges: All inequivalent diagrams (with respect to duality transformations) should be counted with equal weight. Finally, it is indicated that three of the four primitive renormalized loop operators of the theory can be obtained from the formulas of this paper.

I. INTRODUCTION

AN attractive attitude towards the generalized Veneziano model^{1,2} is that it provides one with the Born approximation to a theory of hadrons. The implementation of this idea is seemingly straightforward. One proceeds to factorize the Born term,³ i.e., the N -point tree graphs, thereby deducing the level structure implicit in the model and the Feynman rules

of the theory.⁴ These rules are then to be utilized in the construction of a unitary perturbation series.

The one-loop planar diagrams were in fact easily derived⁵ once it had been demonstrated that the tree graphs were completely factorizable.³ Further progress has, however, been impeded by a series of technical difficulties and by the fundamental problem of renormalization. The technical difficulties relate to the inclusion of twisted vertices and to the removal of the spurious states which are found by naive factorization.

The twisted vertices are necessary for the construction of twisted loop diagrams, some of which correspond to nonplanar Feynman graphs and some of which (the

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¹ G. Veneziano, *Nuovo Cimento* **57A**, 190 (1968).

² C. J. Goebel and B. Sakita, *Phys. Rev. Letters* **22**, 256 (1969); C. H. Mo and T. S. Tsun, *Phys. Letters* **28B**, 485 (1969); K. Bardakci and H. Ruegg, *Phys. Rev.* **181**, 1884 (1969); Z. Koba and H. B. Nielson, *Nucl. Phys.* **B10**, 633 (1969).

³ S. Fubini and G. Veneziano, *Nuovo Cimento* **64A**, 811 (1969); K. Bardakci and S. Mandelstam, *Phys. Rev.* **184**, 1640 (1969).

⁴ S. Fubini, D. Gordon, and G. Veneziano, *Phys. Letters* **29B**, 679 (1969); Y. Nambu, University of Chicago report, 1969 (unpublished).

⁵ K. Bardakci, M. B. Halpern, and J. A. Shapiro, *Phys. Rev.* **185**, 1910 (1969); D. Amati, C. Bouchiat, and J. L. Gervais, *Nuovo Cimento Letters* **2**, 399 (1969).

so-called nonorientable graphs) have no corresponding analog in Feynman theory. Such diagrams are, in general, required in order to satisfy unitarity. Twists can be handled with relative ease by defining a twist operator^{6,7} which takes one from untwisted vertices to twisted ones. The spurious states appear in the factorization of the tree graphs due to linear dependences among the vertices (the so-called Ward identities⁸). There exist, therefore, linear combinations of the states used in the factorization that do not couple to the external scalar particles and give no contribution to the full tree amplitudes. It is possible to explicitly remove them from the model by defining a projection operator onto the subspace of real states.⁸ There is a great advantage to working within this projected subspace, since one can then replace any operator in the theory by another one which has identical matrix elements in the subspace of real states (these are related by the so-called gauge transformations).

The planar one-loop amplitudes exhibit an exponential divergence, which is directly related to the enormous number of states in the theory. Two of us⁹ have shown how to construct a counter term that cancels this divergence, while preserving duality, perturbative unitarity, Regge behavior, and factorization.¹⁰

In this paper, using the techniques described above, we consider the complete set of twisted and untwisted one-loop amplitudes. We carry out the sums over the intermediate states that appear in the loops and reduce the general one-loop amplitude to an integral over elliptic functions. The planar and nonorientable diagrams are renormalized by constructing appropriate counter terms. The nonplanar orientable diagrams are shown to be finite in a certain region of the energy plane but probably require renormalization, too. These amplitudes develop branch points in an energy variable that are unrelated to normal thresholds. We think that these singularities, which are specific to the vacuum channel, are related to the Pomeranchukon, but at the present time we are not sure how to treat them. Finally, we discuss the question of unitarity at the one-loop level. More specifically, we determine the set of N -particle one-loop graphs that is unitary to this order in the coupling constant. It turns out that a unitary result is obtained when all graphs that are not related by a duality transformation are summed with equal weight.

It might appear that the calculation of the one-loop

amplitudes is only the first step in the infinite procedure of evaluating the sum of all diagrams necessary for unitarity. This, however, is not the case in a dual theory. Recently, we have¹¹ analyzed the general N -point diagram for any dual theory and have shown that there are only the four one-particle irreducible diagrams shown in Fig. 1. Any dual Feynman diagram can be constructed, once one knows the four operators which these graphs represent, by attaching appropriate numbers of such graphs onto a tree diagram. Moreover, it is possible that the (formal) sum of all the diagrams required by unitarity can be written explicitly in terms of these four operators. Since three of them are one-loop operators, they can be derived by factorizing the general one-loop planar [Fig. 1(a)], nonorientable [Fig. 1(b)], and orientable nonplanar [Fig. 1(c)] diagrams. In Ref. 10 this procedure was carried out explicitly for the scattering energy operator related to the planar tadpole [Fig. 1(a)]. Furthermore, counterterm operators for Figs. 1(a) and 1(b) can easily be derived.¹⁰ The solution to the one-loop problem contains, therefore, three of the four operators which are required to construct the full unitary theory.

Finally, we are considering a model with no internal degrees of freedom, and with several undesirable properties (ghosts, negative-intercept trajectories, lack of fermions, etc.). There are indications^{12,13} that more realistic models can be constructed by suitable alterations of the Born term. It is unlikely, however, that these will change in any fundamental way the dynamics of the loop amplitudes considered in this paper.

In Sec. II we review the algebraic methods necessary for the calculation of loop diagrams, especially the methods used for treating twists and spurious states and projections. The sum over intermediate states in the general loop integral is explicitly carried out. Section III is devoted to a short review of the planar diagrams, and their renormalization. Section IV deals with the nonorientable diagrams. These are calculated and renormalized. In Sec. V we calculate the nonplanar orientable diagrams. It is shown that naive application of the Feynman rules involves an infinite counting, which can, however, be easily remedied. The new singularity generated by this diagram are discussed. In Sec. VI we discuss the unitarity of the N -point one-loop diagrams. The Appendix contains a discussion of the elliptic functions that appear in the loop integrals.

II. PROJECTIONS, TWISTS, AND DIAGONALIZATION OF TRACES

In this section we review the algebraic methods that have been developed for calculating the N -point or

⁶ L. Caneschi, A. Schwimmer, and G. Veneziano, *Phys. Letters* **30B**, 351 (1969).

⁷ D. Amati, M. Le Bellac, and D. Olive, *Nuovo Cimento* **66A**, 815 (1970); **66A**, 831 (1970); F. Gliozzi, *Nuovo Cimento Letters* **2**, 846 (1969); C. B. Chiu, S. Matsuda, and C. Rebbi, *Phys. Rev. Letters* **23**, 1526 (1969).

⁸ C. B. Thorn, *Phys. Rev. D* **1**, 1693 (1970); M. Kaku and C. B. Thorn, *ibid.* **1**, 2860 (1970); R. Brower and J. Weis, *Nuovo Cimento Letters* **3**, 285 (1970).

⁹ A. Neveu and J. Scherk, *Phys. Rev. D* **1**, 2355 (1970); G. Frye and L. Susskind, *Phys. Letters* **31B**, 537 (1970).

¹⁰ T. H. Burnett, D. J. Gross, A. Neveu, J. Scherk, and J. H. Schwarz, *Phys. Letters* **32B**, 115 (1970).

¹¹ D. J. Gross, A. Neveu, J. Scherk, and J. H. Schwarz, *Phys. Letters* **31B**, 592 (1970).

¹² M. A. Virasoro, *Phys. Rev. D* **1**, 2933 (1970).

¹³ K. Bardakci and M. Halpern, *Phys. Rev. Letters* **24**, 1014 (1970).

loop integral with an arbitrary number of twisted lines. We also show explicitly the algebra required to remove the contribution of spurious states that do not appear in the primitive N -point functions. Almost all the techniques and results of this section can be found in Refs. 4–8, especially the work of Kaku and Thorn. It should be helpful, however, to have the salient facts all collected in one place.

We begin by recalling that the level structure of the dual-resonance model is described by vectors in a Hilbert space generated by an infinite number of four-vector creation operators, $a_n^{\dagger\mu}$, $n=1, 2, \dots$.⁴ The propagator for momentum p is given by the operator

$$D(R, p) = \int_0^1 dx (1-x)^{\alpha_0-1} x^{R-\alpha(p^2)-1}, \quad (2.1)$$

where

$$R = \sum_{n=1}^{\infty} n a_n^{\dagger} \cdot a_n$$

and one has linear trajectory functions

$$\alpha(s) = \alpha_0 + \frac{1}{2}s.$$

The vertex operator for coupling an on-shell scalar state of momentum k [with $\alpha(k^2=m^2)=0$] is

$$V(k) = g \exp\left(\sum_{n=1}^{\infty} \frac{k \cdot a_n^{\dagger}}{\sqrt{n}}\right) \exp\left(\sum_{n=1}^{\infty} \frac{k \cdot a_n}{\sqrt{n}}\right). \quad (2.2)$$

These operators are sufficient to reproduce the standard N -point functions in any multiperipheral configuration.

We now introduce the three additional operators ($p^2=m^2$ in our metric)

$$\begin{aligned} L_0(p) &= R - \frac{1}{2}p^2, \\ L_+(p) &= p \cdot a_1^{\dagger} + \sum_{n=1}^{\infty} [n(n+1)]^{1/2} a_{n+1}^{\dagger} \cdot a_n, \\ L_-(p) &= -p \cdot a_1 + \sum_{n=1}^{\infty} [n(n+1)]^{1/2} a_n^{\dagger} \cdot a_{n+1}. \end{aligned} \quad (2.3)$$

These three operators generate the algebra $SU(1,1)$ ⁷:

$$[L_0, L_{\pm}] = \pm L_{\pm}, \quad [L_+, L_-] = -2L_0. \quad (2.4)$$

Next, define

$$\begin{aligned} A(p) &\equiv A = L_0(p) - L_-(p), \\ A^{\dagger}(-p) &\equiv A^{\dagger} = L_0(p) - L_+(p). \end{aligned}$$

$A^{\dagger}(p)$ has been constructed in such a way as to create, when applied to any state in the Hilbert space, "spurious" states that do not couple to any number of external scalar particles. Therefore, one has the option of removing such states from the theory.¹⁴ This is done

¹⁴ This is highly desirable since otherwise one must enlarge the Hilbert space, by introducing additional scalar modes, in order to achieve factorization of the twisted vertices (see Amati *et al.*, Ref. 7). With the projections one never needs to introduce additional scalar modes.

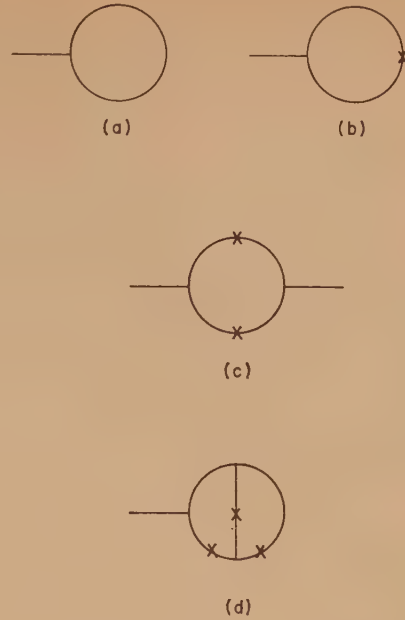


FIG. 1. The four primitive graphs of the dual-resonance theory.

by introducing the projection operator⁸

$$P = I - (A^{\dagger} - \alpha_0)[A(A^{\dagger} - \alpha_0)]^{-1}A, \quad (2.5)$$

which satisfies

$$\begin{aligned} D(R, p)P(p) &= P^{\dagger}(-p)D(R, p), \\ P(p+k)V(k) &= V(k)P^{\dagger}(-p). \end{aligned} \quad (2.6)$$

The projected propagator $P^{\dagger}(-p)D(R, p)P(p)$ is Hermitian and contains states which have linearly independent couplings to external scalar particles.

The twist operator of Caneschi, Schwimmer, and Veneziano⁶ is given by

$$\Omega = (-1)^R e^{-L_+} = e^{(1-\beta)L_+} (-1)^R e^{-\beta L_+}, \quad (2.7)$$

where the second form is made possible by the general rule

$$x^{L_0} f(L_+) = f(xL_+) x^{L_0}. \quad (2.8)$$

This twist is unique only up to a gauge transformation generated by A . It is shown in Ref. 7 that the choice

$$\Theta(z) = \Omega(1-z)^A = (1-z)^{-A}\Omega, \quad (2.9)$$

where z is the integration variable of the adjacent propagator (Θ is brought inside the integral), has the desirable Hermiticity property

$$z^R \Theta(z) = \Theta^{\dagger}(z) z^R. \quad (2.10)$$

The last formula guarantees that the result of a calculation does not depend on the placement of dots; i.e., twisting on both sides of a propagator is equivalent to not twisting at all.

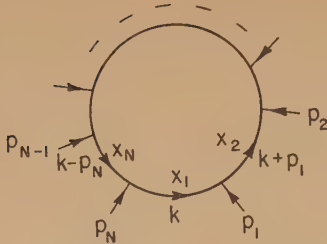


FIG. 2. General one-loop diagram in the configuration where the external lines are attached directly to the loop.

Recording two more useful identities proved by Thorn,⁸

$$A(p+k)V(k)D(R,p)=V(k)D(R+1,p)A(p), \quad (2.11)$$

$$A(p)\Omega(p)=-\Omega(p)A(p), \quad (2.12)$$

we may proceed to the calculation of traces. The problem is to calculate a diagram of the form shown in Fig. 2 in which an arbitrary number of the internal lines are twisted. The corresponding integral (aside from modifications to be discussed in subsequent sections) is

$$\int d^4k \operatorname{Tr}\{V(p_1)P^\dagger(k+p_1)D(R,k+p_1)[1+\Theta(k+p_1)] \\ \times P(-k-p_1)V(p_2)P^\dagger(k+p_1+p_2)\cdots V(p_N) \\ \times P^\dagger(k)D(R,k)[1+\Theta(k)]P(-k)\}.$$

We have introduced a factor $1+\Theta$ for each line. What we really mean is to choose 1 or Θ for each line depending on which lines are to be twisted. The commutation and projection properties of P enable one to replace Θ 's by Ω 's and drop all but one of the projections in the trace, leaving us with

$$\operatorname{Tr}\{V(p_1)D(R,k+p_1)[1+\Omega(k+p_1)]V(p_2)\cdots V(p_N) \\ \times L(R,k)[1+\Omega(k)]P(-k)\}.$$

Finally, bringing the A on the right of $1-P$ around through the trace gives⁸

$$\operatorname{Tr}\{V(p_1)D(R,k+p_1)[1+\Omega(k+p_1)] \\ \times V(p_2)\cdots V(p_N)L(R,k)[1+\Omega(k)]\} \\ -\operatorname{Tr}\{V(p_1)D(R+1,k+p_1)[1-\Omega(k+p_1)] \\ \times V(p_2)\cdots V(p_N)D(R+1,k)[1-\Omega(k)]\}.$$

Therefore, we conclude that for a particular choice of M twists one has

$$[1-(-1)^M w] \operatorname{Tr}\{V(p_1)D(R,k+p_1)[1 \text{ or } \Omega(k+p_1)] \\ \times V(p_2)\cdots V(p_N)L(R,k)[1 \text{ or } \Omega(k)]\}, \quad (2.13)$$

where

$$w = \prod_{i=1}^N x_i,$$

and x_i is the integration variable of the i th propagator.

The factor $1\pm w$ is understood, of course, to be brought inside the propagator integrals. The problem of the projection is thus completely solved and the next question is how to calculate the traces.

The trace calculation is completely straightforward using the properties of coherent states, when only vertices and diagonal operators appear.⁵ When one or more twists are also present, such a form is not manifest. Therefore, for the configuration of Fig. 3, we now show how to reduce the trace to a form which is as simple as the one without twists. The restriction $M \geq 1$ adjacent twisted lines is strictly for convenience; the generalization will be made evident. Dropping the nonoperator parts of the trace and writing the twisted operators in the second form of Eq. (2.7), we wish to calculate

$$\operatorname{Tr}\{x_1^R \Omega_{\beta_1} V x_2^R \Omega_{\beta_2} \cdots x_M^R \Omega_{\beta_M} V y_1^R V \cdots y_N^R V\}.$$

The key equation required to simplify the algebra

$$e^{\beta L_+} = (1+\beta)^{L_0} (1+\beta)^{L_+ - L_0}. \quad (2.14)$$

This relation is proved by first showing

$$F(1+\beta) = (1+\beta)^{L_+ - L_0} = e^{L_+} (1+\beta)^{-L_0} e^{-L_+}, \quad (2.15)$$

which is true because both sides have the group property $F(\gamma_1)F(\gamma_2)=F(\gamma_1\gamma_2)$, and the infinitesimal operators are easily seen to agree. Equation (2.15) is then combined with

$$(1+\beta)^{L_0} e^{L_+} = e^{(1+\beta)L_+} (1+\beta)^{L_0}, \quad (2.16)$$

which is a special case of Eq. (2.8), to give Eq. (2.14). The additional relations

$$(1-\beta)^{\alpha_0 - A^\dagger(-p-k)} V(k) = V(k) (1-\beta)^{-A^\dagger(-p)} \quad (2.17)$$

and

$$e^{-\beta L_+} x^R = x^R e^{-\beta L_+} x \quad (2.18)$$

provide all the algebraic tools needed.

Equations (2.7), (2.14), (2.17), and (2.18) may be

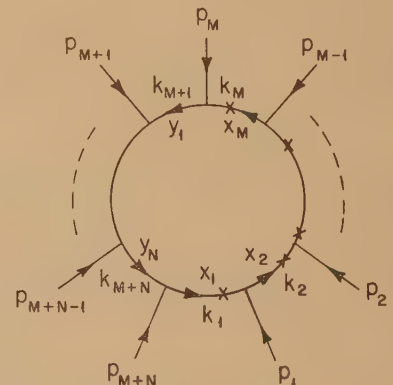


FIG. 3. One-loop diagram having M adjacent twisted internal lines and N adjacent untwisted internal lines.

combined to show that

$$\Omega_\beta V x^R = e^{(1-\beta)L_+} (-1)^R (1-\beta)^{L_0-\alpha_0} V (1-\beta)^{-L_0} \times x^R e^{-\beta L_+/x}. \quad (2.19)$$

This relation is used repeatedly for every twist in the trace. The β 's, which until now were arbitrary, are then chosen to cancel the L_+ 's occurring in the exponentials. This cancellation requires that

$$\frac{\beta_1}{x_2} = 1 - \beta_2, \quad \frac{\beta_2}{x_3} = 1 - \beta_3, \dots, \\ \frac{\beta_{M-1}}{x_M} = 1 - \beta_M, \quad \frac{\beta_M}{y_1 y_2 \dots y_N x_1} = 1 - \beta_1.$$

It then follows that the operator part of the remaining trace is

$$\text{Tr}\{(-u_1)^R V(-u_2)^R \dots (-u_M)^R \times V v_1^R V \dots v_N^R V\}, \quad (2.20)$$

where

$$u_1 = \frac{\beta_M}{y_1 y_2 \dots y_N - \beta_M}, \quad u_2 = \frac{\beta_1}{1 - \beta_1}, \dots, \\ u_M = \frac{\beta_{M-1}}{1 - \beta_{M-1}}, \quad (2.21)$$

and

$$v_1 = \frac{y_1 - \beta_M}{1 - \beta_M}, \quad v_2 = \frac{y_1 y_2 - \beta_M}{y_1 - \beta_M}, \dots, \\ v_N = \frac{y_1 y_2 \dots y_N - \beta_M}{y_1 y_2 \dots y_{N-1} - \beta_M}. \quad (2.22)$$

Eliminating the β 's between Eqs. (2.21) and (2.22) yields

$$x_1 = \frac{u_1(1+u_2)}{1+u_1}, \quad x_2 = \frac{u_2(1+u_3)}{1+u_2}, \dots, \\ x_{M-1} = \frac{u_{M-1}(1+u_M)}{1+u_{M-1}}, \quad (2.23) \\ x_M = \frac{u_M(1+v_1 v_2 \dots v_N u_1)}{1+u_M},$$

and

$$y_1 = \frac{v_1 + v_1 v_2 \dots v_N u_1}{1 + v_1 v_2 \dots v_N u_1}, \quad (2.24) \\ y_2 = \frac{v_2 + v_2 v_3 \dots v_N u_1}{1 + v_2 v_3 \dots v_N u_1}, \dots, \quad y_N = \frac{v_N + v_N u_1}{1 + u_1}.$$

Note that the product of u 's and v 's is the same as the

product of x 's and y 's,

$$w = \prod_{i=1}^M x_i \prod_{j=1}^N y_j = \prod_{i=1}^M u_i \prod_{j=1}^N v_j.$$

The Jacobian for changing from x and y integrations to u and v integrations is

$$\prod_{i=1}^M dx_i \prod_{j=1}^N dy_j = [1 - (-1)^M w] \\ \times \prod_{i=1}^M \frac{du_i}{1 + u_i} \prod_{j=1}^N \frac{dv_j}{1 + v_j v_{j+1} \dots v_N u_1}.$$

Finally, collecting factors that have been dropped along the way, we may assemble the formula for the loop integral with M adjacent twists,

$$F_{N,M} = \int d^4 k \int \left[\prod_{i=1}^M du_i u_i^{-\alpha(k_i^2)-1} \right] \left[\prod_{j=1}^N dv_j v_j^{-\alpha(k_j^2+M^2)-1} \right] \\ \times [1 - (-1)^M w]^2 [(1 - u_M v_1 v_2 \dots v_N u_1) \\ \times \prod_{i=1}^{M-1} (1 - u_i u_{i+1}) \prod_{j=1}^N (1 - v_j)]^{\alpha_0-1} \\ \times \text{Tr}\{(-u_1)^R V(p_1)(-u_2)^R \dots (-u_M)^R \\ \times V(p_M) v_1^R V(p_{M+1}) \dots v_N^R V(p_{M+N})\}. \quad (2.25)$$

We have now accomplished the goal of rewriting the loop integral in a form for which the trace calculations are no worse in twisted configurations than in untwisted ones. Superficially, at least, the upper limits of the u and v integrations correspond to the zeros of the factors that appear to the power $\alpha_0 - 1$. This rule will be modified in later sections.

The general rule for a one-loop diagram in the configuration of Fig. 3 with M arbitrarily placed twisted lines is now quite evident. (1) Jacobians and projections contribute a factor $[1 - (-1)^M w]^2$ except when $M=0$, in which case the factor is just $1-w$. (2) Each line gives a factor $u^{-\alpha(k^2)-1}$ (where u is a twisted line of momentum k) or $v^{-\alpha(k^2)-1}$ (where v is an untwisted line of momentum k). (3) There is a trace in which the lines and vertices appear in cyclic order with an untwisted line giving v^R , and a twisted line giving $(-u)^R$. (4) The following factors appear to the power $\alpha_0 - 1$: a factor $(1-v_i)$ for each untwisted line v_i , and a factor of the form $(1-u_k v_i v_{i+1} \dots v_{i+m} u_{k+1})$, where u_k and u_{k+1} refer to two successive twisted lines with $m+1$ intervening untwisted lines, for each twisted line u_k . In the particular case of just one twisted line, the last factor becomes $1-uw$. (5) Finally, the u 's and v 's are integrated over the region in which they are positive and in which the factors given above are also positive. This rule will be modified in Sec. V in the case where M is even and nonzero.

III. PLANAR DIAGRAMS

In order to deal with the simplest case first, we consider the N -point loop with no twists. The rules of Sec. II give for this case

$$F_N = \int d^4k \int_0^1 \prod_{i=1}^N [dx_i (1-x_i)^{\alpha_0-1} x_i^{-\alpha(k_i^2)-1}] (1-w) T, \quad (3.1)$$

where, using the techniques of Amati *et al.*⁵ to evaluate

the trace,

$$\begin{aligned} \mathcal{T} &= \text{Tr}\{x_1^R V(p_1) x_2^R \cdots x_N^R V(p_N)\} \\ &= g^N [f(w)]^{-4} \prod_{i,j=1}^N \prod_{n=1}^{\infty} \exp \left[\frac{p_i \cdot p_j C_{ji}^n}{n(1-w^n)} \right], \end{aligned} \quad (3.2)$$

$$f(w) = \prod_{n=1}^{\infty} (1-w^n),$$

$$C_{ji} = x_{i+1} x_{i+2} \cdots x_j \text{ (in cyclic order).}$$

Thus, since $C_{ii} = w$ and $C_{ij} = w/C_{ji}$ for $i \neq j$,

$$\begin{aligned} T &= g^N [f(w)]^{-4} \prod_{n=1}^{\infty} \left\{ \prod_{1 \leq i < j \leq N} \exp \left[\frac{C_{ji}^n + (w/C_{ji})^n}{n(1-w^n)} \right] \prod_{i=1}^N \exp \left[\frac{w^n}{n(1-w^n)} \right] \right\} \\ &= g^N [f(w)]^{-4} \prod_{n=1}^{\infty} \prod_{1 \leq i < j \leq N} \exp \left[\frac{C_{ji}^n + (w/C_{ji})^n - 2w^n}{n(1-w^n)} \right], \end{aligned}$$

where the last step uses momentum conservation. We can now recognize the infinite products as Jacobi θ functions. Using

$$\theta_1 \left(\frac{\ln x}{2\pi i} \middle| \frac{\ln w}{2\pi i} \right) = i x^{-1/2} f(w) w^{1/8} \prod_{n=0}^{\infty} (1 - x w^n) (1 - w^{n+1}/x),$$

we find that

$$\begin{aligned} T &= g^N [f(w)]^{-4} \prod_{1 \leq i < j \leq N} \left[-2\pi i C_{ji}^{1/2} \theta_1 \left(\frac{\ln C_{ji}}{2\pi i} \middle| \frac{\ln w}{2\pi i} \right) / \theta_1' \left(0 \middle| \frac{\ln w}{2\pi i} \right) \right]^{p_i \cdot p_j} \\ &= g^N [f(w)]^{-4} \left(\frac{w}{x} \right)^{-\alpha_0} \left(\prod_{i=2}^{N-1} x_{i+1}^{-\alpha_1 i} \right) \prod_{1 \leq i < j \leq N} \left[-2\pi i \theta_1 \left(\frac{\ln C_{ji}}{2\pi i} \middle| \frac{\ln w}{2\pi i} \right) / \theta_1' \left(0 \middle| \frac{\ln w}{2\pi i} \right) \right]^{-p_i \cdot p_j}, \end{aligned} \quad (3.3)$$

where the derivatives refer to the first variable of the θ_1 function, a standard notation.

The momentum integration is now required for putting F_N into the desired form. Following tradition, we do not discuss the validity of the Wick rotation that is needed.

$$\begin{aligned} i) \tilde{\mathcal{M}} &= \int d^4k \prod_{i=1}^N x_i^{-\alpha(k_i^2)-1} \\ &= \frac{4\pi^2}{\ln^2 w} w^{-\frac{1}{2}\alpha_0-1} \exp \left(\frac{\alpha_0}{2 \ln w} \sum_{i=1}^N \ln^2(x_i) \right) \exp \left(- \sum_{1 \leq i < j \leq N-1} \alpha_{ij} \frac{\ln x_i \ln x_{j+1}}{\ln w} \right) \\ &= \frac{4\pi^2}{w \ln^2 w} x_1^{-\alpha_0} \left(\prod_{i=2}^{N-1} x_{i+1}^{-\alpha_1 i} \right) \prod_{1 \leq i < j \leq N} \exp \left(- \frac{p_i p_j \ln^2 C_{ji}}{2 \ln w} \right). \end{aligned} \quad (3.4)$$

To carry out the last step in the algebra of (3.4), it is useful to know that momentum conservation and the mass-shell constraint allow one to prove the general rule

$$\begin{aligned} &\prod_{\substack{i < j \leq N-1}} \left(\frac{B_{i,j} B_{i-1,j+1}}{B_{i,j+1} B_{i-1,j}} \right)^{-\alpha_{ij}} \\ &= \left(\prod_{i=1}^N B_{i-1,i}^{\alpha_0} \right) \left(\prod_{1 \leq i < j \leq N} B_{i,j}^{-p_i \cdot p_j} \right), \end{aligned}$$

where one defines $B_{0,j} \equiv B_{j,N}$. [In using this rule in (3.4), the $i=1$ terms require special care.] Inserting

(3.3) and (3.4) into (3.1), we obtain

$$\begin{aligned} F_N &= 4\pi^2 g^N \int \prod_{i=1}^N [dx_i (1-x_i)^{\alpha_0-1}] \\ &\times \frac{w^{-\alpha_0-1} [f(w)]^{-4}}{\ln^2 w} \prod_{1 \leq i < j \leq N} [\psi(C_{ji})]^{-p_i \cdot p_j}, \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} \psi(x) &= \psi(x, w) = -2\pi i \\ &\times \exp \left(\frac{\ln^2 x}{2 \ln w} \right) \theta_1 \left(\frac{\ln x}{2\pi i} \middle| \frac{\ln w}{2\pi i} \right) / \theta_1' \left(0 \middle| \frac{\ln w}{2\pi i} \right). \end{aligned} \quad (3.6)$$

Several properties of $\psi(x)$ are discussed in the Appendix. In particular, we note the property $\psi(x) = \psi(w/x)$. This means that for a particular term with exponent $p_i \cdot p_j$ the argument of the ψ function is the product of the x variables that appear between the two momenta, and it does not matter on which side these x 's are chosen. One immediate consequence of this invariance is the cyclic symmetry of F_N in the external momenta.

As has already been exhaustively discussed in the literature, F_N has an exponential divergence as $w \rightarrow 1$, which is entirely attributable to the factor $[f(w)]^{-4}$. A useful formula for displaying this divergence, which is a corollary of the θ -function transformations discussed in the Appendix, is

$$\frac{4\pi^2}{\ln^2 w} [f(w)]^{-4} = w^{1/6} q^{-1/3} [f(q^2)]^{-4}, \quad (3.7)$$

where

$$q = \exp(2\pi^2/\ln w).$$

It is shown in the Appendix that

$$\tilde{\psi}(x) = -\frac{\ln w}{\pi} \sin\left(\frac{\pi \ln x}{\ln w}\right) \quad (3.8)$$

differs from $\psi(x)$ by $O(q^2)$ as $w \rightarrow 1$. Therefore, if we construct $\tilde{F}_N$ by replacing ψ by $\tilde{\psi}$ in (3.5), the difference $F_N - \tilde{F}_N$ is finite. The duality and analyticity properties of $\tilde{F}_N$ that allow it to be interpreted as an acceptable counterterm have been discussed in Ref. 9. The demonstration that it can be consistently factorized was given in Ref. 10. We emphasize that $\tilde{F}_N$ is not unique, however. For example, it is possible, and perhaps more natural, to replace $f(q^2)$ by unity in (3.7), obtaining

$$\tilde{F}_N = g^N \int_0^1 \prod_{i=1}^N [dx_i (1-x_i)^{\alpha_0-1}] \times w^{-\alpha_0-5/6} q^{-1/3} \prod_{i < j} \tilde{\psi}(C_{ji})^{-p_i \cdot p_j}. \quad (3.9)$$

With this definition, the integrand involves elementary functions only.

IV. NONORIENTABLE DIAGRAMS

Diagrams containing loops with an odd number of twists are called nonorientable. This is because of the dual diagram surface.¹⁵ If one draws such a loop in terms of quark lines, then one finds two quarks circulating around the nonorientable loop in the same direction. Restricting to mesons, such diagrams have no correspondence with the quark model. This had led the authors of Ref. 16 to suggest that when internal quantum numbers are added to the theory in a suitable

fashion, these diagrams give no contribution. Here we restrict ourselves to a theory of external scalar mesons, with no internal symmetry, in which case the nonorientable diagrams are manifestly nonvanishing. Also, as we shall see in Sec. VI, these diagrams are required for unitarity.

The trace calculation and momentum integrations are performed just as in Sec. III. For concreteness, let us consider the case of M adjacent twists discussed in Sec. II. We then have

$$F_{N,M} = 4\pi^2 g^{N+M} \int \prod_{i=1}^N dv_i \prod_{j=1}^M du_j \frac{(1+w)^2 w^{-\alpha_0-1}}{\ln^2 w} \times [f(-w)]^{-4} [(1-u_M v_1 \cdots v_N u_1) \times \prod_{j=1}^{M-1} (1-u_j u_{j+1}) \prod_{i=1}^N (1-v_i)]^{\alpha_0-1} \times \prod_{i < j} [\psi_N(C_{ji}) \text{ or } \psi_{NT}(C_{ji})]^{-p_i \cdot p_j}. \quad (4.1)$$

The functions ψ_N and ψ_{NT} , which differ from ψ by some strategic minus signs, are discussed in the Appendix. The subscript N denotes "nonorientable" and the subscript T denotes "twisted." The rule for choosing the appropriate function is the following: C_{ji} denotes the product of the u 's and v 's appearing between p_i and p_j . If there are an even number of u 's (twisted lines) use ψ_N , whereas if there are an odd number use ψ_{NT} . As we are considering diagrams with an odd number of twists, the identity $\psi_N(x) = \psi_{NT}(w/x)$, proved in the Appendix, ensures that the result does not depend on which of the two possible products of u 's and v 's we choose.

It is also shown in the Appendix that the functions

$$\tilde{\psi}_N = \frac{-2 \ln w}{\pi} \sin\left(\frac{\pi \ln x}{2 \ln w}\right) \quad (4.2a)$$

and

$$\tilde{\psi}_{NT} = \frac{-2 \ln w}{\pi} \cos\left(\frac{\pi \ln x}{2 \ln w}\right) \quad (4.2b)$$

differ from ψ_N and ψ_{NT} , as $w \rightarrow 1$, by $O(q_N^2)$, where

$$q_N = -i \exp(\pi^2/2 \ln w).$$

The term $[f(-w)]^{-4}$ diverges near $w \rightarrow 1$ like $q_N^{-1/3}$. Therefore, just as in the planar case, we construct a counterterm $\tilde{F}_{N,M}$ that cancels the divergence of $F_{N,M}$ by replacing ψ_N and ψ_{NT} by $\tilde{\psi}_N$ and $\tilde{\psi}_{NT}$, respectively. This rule is also satisfactory in its duality and analyticity properties. This nonorientable counterterm, just as the planar one, is nonunique; one can easily add to it an additional finite counterterm.

The nonorientable loop integrals have an additional interesting property that will prove quite useful in the discussion of unitarity in Sec. VI. Namely, there is a natural way to decompose the integration region into pieces corresponding to different diagrams. We now

¹⁵ K. Kikkawa, S. Klein, B. Sakita, and M. Virasoro, Phys. Rev. D **1**, 3258 (1970).

¹⁶ M. B. Halpern, S. Klein, and J. Shapiro, Phys. Rev. **188**, 2378 (1969).

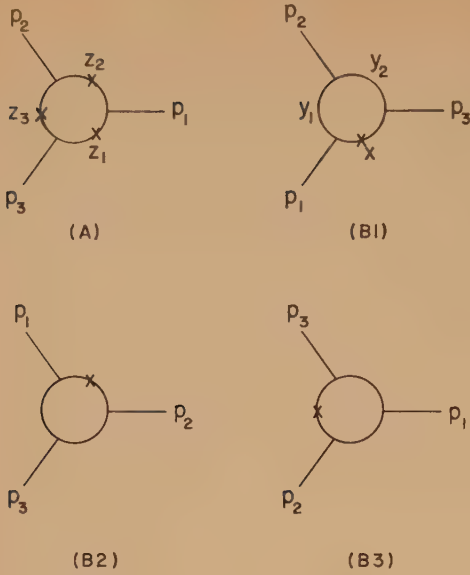


FIG. 4. Equivalent forms for a particular nonorientable loop diagram.

show how this works for a particular example and then state the general rule without further proof.

Consider the diagram labeled configuration A in Fig. 4. According to the rules we have formulated, the integral associated with this diagram is

$$F_A = 4\pi^2 g^3 \int dz_1 dz_2 dz_3 \frac{(1+w)^2 [f(-w)]^{-4}}{\ln^2 w} w^{-\alpha_0-1} \\ \times [(1-z_1 z_2)(1-z_2 z_3)(1-z_3 z_1)]^{\alpha_0-1} \\ \times \psi_N(z_1 z_2)^{-p_1 \cdot p_2} \psi_{NT}(z_1)^{-p_1 \cdot p_3} \psi_N(z_1 z_2)^{-p_2 \cdot p_3}. \quad (4.3)$$

Now consider the diagrams labeled configurations B_1 , B_2 , and B_3 in Fig. 4. The diagram B_i can be obtained from configuration A by dualizing the line z_i . Therefore, these diagrams should give the same integrals as A . To check this property consider B_1 ,

$$F_{B1} = 4\pi^2 g^3 \int dx dy_1 dy_2 \frac{(1+w)^2 [f(-w)]^{-4}}{\ln^2 w} w^{-\alpha_0-1} \\ \times [(1-y_1)(1-y_2)(1-wx)]^{\alpha_0-1} \\ \times \psi_N(y_1)^{-p_1 \cdot p_2} \psi_{NT}(x)^{-p_1 \cdot p_3} \psi_N(y_2)^{-p_2 \cdot p_3}. \quad (4.4)$$

It is obvious that the equality of F_A and F_{B1} (as well as their counterterms) is demonstrated by the change of variables

$$x = 1/z_1, \quad y_1 = z_1 z_3, \quad y_2 = z_1 z_2. \quad (4.5)$$

This requires the identity $\psi_{NT}(x) = \psi_{NT}(1/x)$, proved in the Appendix. The striking fact is that the integration region of the full expression,

$$z_1, z_2, z_3 \geq 0; \quad z_1 z_2, z_2 z_3, z_3 z_1 \leq 1,$$

can be decomposed into the four pieces

$$A: \quad 0 \leq z_1, z_2, z_3 \leq 1,$$

$$B_i: \quad 0 \leq 1/z_i, \quad z_i z_{i+1}, \quad z_i z_{i+1} \leq 1, \quad i=1, 2, 3$$

which are naturally identified with the four diagrams of Fig. 4 in the way indicated. In other words, the full duality integral can be decomposed into a sum of the four pieces obtained by taking each of the four diagrams with its natural integration variables restricted to the range 0 to 1.

The generalization of the above result to N external lines is as follows: Corresponding to a particular nonorientable one-loop diagram, there are 2^{N-1} equivalent configurations obtainable by duality transformations provided we only consider diagrams with the external lines connected directly to the loop, i.e., diagrams of the general form considered in Sec. II. The integral corresponding to any one of these diagrams is the same, but can be expressed as a sum of 2^{N-1} terms corresponding to the various configurations with the integrals in each case restricted to the "unit cube," i.e., with the integration for each of the natural variables running from 0 to 1. The advantage of this decomposition is that each of the "cube-restricted diagrams" only has normal-threshold cuts in the channels that explicitly display them. This result is extremely useful when we examine the unitarity of the loops in Sec. VI.

V. ORIENTABLE NONPLANAR DIAGRAMS

Diagrams with an even number of twists (greater than zero) have some very startling features. In order to study these features we begin, as before, by writing down the general form of the orientable one-loop integral in the case of M adjacent twists,

$$F_{N,M} = 4\pi^2 g^{N+M} \int \left(\prod_{i=1}^N dv_i \prod_{j=1}^M du_j \right) \\ \times (1-w)^2 \frac{f(w)^{-4}}{\ln^2 w} w^{-\alpha_0-1} [(1-u_M v_1 \cdots v_N u_1)] \\ \times \prod_{j=1}^{M-1} (1-u_j u_{j+1}) \prod_{i=1}^N (1-v_i)^{\alpha_0-1} \\ \times \prod_{i < j} [\psi(C_{ji}) \text{ or } \psi_T(C_{ji})]^{-p_i \cdot p_j}. \quad (5.1)$$

ψ is used when an even number of twisted lines are included between p_i and p_j and ψ_T (another function described in the Appendix) is used for an odd number. Once again, the symmetries $\psi(x) = \psi(w/x)$ and $\psi_T(x) = \psi_T(w/x)$ (shown in the Appendix) prevent the formula from depending on which set of included lines is chosen.

The new problems associated with these diagrams can be illustrated by the example shown in Fig. 5(a).

The corresponding integral is

$$F_a = 4\pi^2 g^4 \int d^4x (1-w)^2 \frac{[f(w)]^{-4}}{\ln^2 w} w^{-\alpha_0-1} \\ \times [(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_1)]^{\alpha_0-1} \\ \times \psi_T(x_2)^{-p_1 \cdot p_2} \psi(x_2x_3)^{-p_1 \cdot p_3} \psi_T(x_1)^{-p_1 \cdot p_4} \\ \times \psi_T(x_3)^{-p_2 \cdot p_3} \psi(x_3x_4)^{-p_2 \cdot p_4} \psi_T(x_4)^{-p_3 \cdot p_4}. \quad (5.2)$$

The integral F_a is invariant under two types of variable changes:

$$x_1' = x_1 w, \quad x_2' = x_2/w, \quad x_3' = x_3 w, \quad x_4' = x_4/w \quad (5.3)$$

and

$$x_1' = 1/x_1, \quad x_2' = w/x_2, \quad x_3' = 1/x_3, \quad x_4' = x_4/w.$$

The additional symmetry $\psi_T(x) = \psi_T(1/x)$ is shown in the Appendix. This invariance is important because these transformations allow one to see that the "naive" integration region

$$x_i > 0, \quad x_i x_{i+1} < 1, \quad i = 1, 2, 3, 4$$

can be divided into an infinite number of cells, each of which contributes equally. (Hence the integrals are divergent in a rather trivial way.) The source of this circumstance can be traced to the fact that a sequence of duality transformations can be applied to Fig. 3(a) so as to return it to the original configuration while at the same time inducing one of the changes of variables of (5.3). The upshot of this is that the absorptive parts one might naively have hoped to incorporate in these integrals are in fact counted over and over again an infinite number of times. (This has also been noted by Kaku and Thorn.⁸) The integral with the correct absorptive parts can be obtained by restricting the integration to a single one of the cells.

Consider now the configuration of Fig. 5(b), which can be obtained from Fig. 5(a) by dualizing the line x_2 . The corresponding integral is given by

$$F_b = 4\pi^2 g^4 \int d^2u d^2v (1-w)^2 \frac{[f(w)]^{-4}}{\ln^2 w} w^{-\alpha_0-1} \\ \times [(1-u_1)(1-u_2)(1-v_1u_2v_2)(1-v_2u_1v_1)]^{\alpha_0-1} \\ \times \psi_T(v_1)^{-p_1 \cdot p_2} \psi(u_2)^{-p_1 \cdot p_3} \psi_T(u_1v_1)^{-p_1 \cdot p_4} \\ \times \psi_T(v_1u_2)^{-p_2 \cdot p_3} \psi(u_1)^{-p_2 \cdot p_4} \psi_T(v_2)^{-p_3 \cdot p_4}. \quad (5.4)$$

Using the relation $\psi_T(x) = \psi_T(1/x)$, the equality of (5.2) and (5.4) is made manifest through the change of variables

$$x_1 = u_1v_1, \quad x_2 = 1/v_1, \quad x_3 = v_1u_2, \quad x_4 = v_2. \quad (5.5)$$

A similar transformation takes one to the configuration of Fig. 5(c). Now suppose we restrict the integration for each of the three diagrams to a unit (four-dimensional)

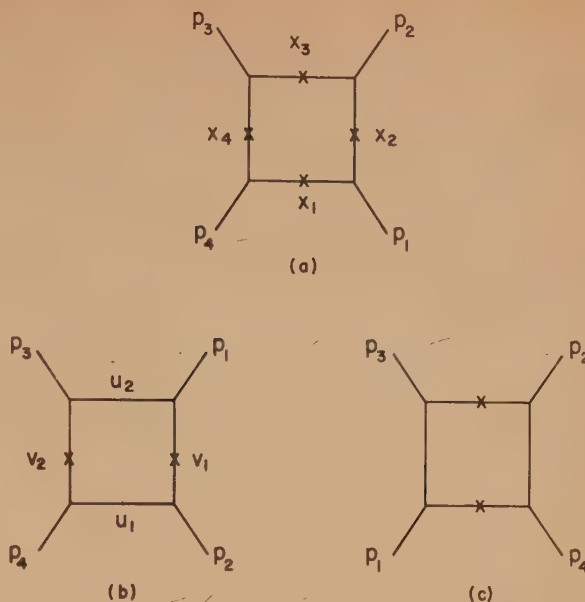


FIG. 5. Equivalent forms for a particular orientable nonplanar loop diagram.

cube. It is then evident that the three regions so defined are disjoint when expressed in terms of the x variables. Furthermore, they remain disjoint even when one allows for the transformations of (5.3). In fact, taken together these three regions precisely define a cell which, with its images obtained by repeated application of (5.3), fill up the original integration region. The general rule, identical to the nonorientable case (and trivially to the planar as well), is to take each of the different loop configurations (of the form of Fig. 2) that can be reached by duality transformations of a particular diagram and restrict the integrations to the cubes defined by having the "natural" variables go from 0 to 1. Diagrams related by duality transformations can then be combined into one with an appropriately enlarged integration region, which in fact is the basic cell corresponding to the invariances of the type shown in (5.3).

We now turn to the question of renormalization. We recall that in the examples of Secs. III and IV the integrals contained an exponential divergence for $w \rightarrow 1$, which was entirely due to the factor $[f(\pm w)]^{-4}$. The properties of ψ , ψ_N , and ψ_{NT} turned out to make possible the subtraction of an appropriately defined counterterm so as to obtain a well-defined and physically sensible result. The orientable nonplanar diagrams which we are considering now are the only one-loop diagrams that contain the function ψ_T . As is shown in the Appendix, this function differs in a fundamental way from the other three. Namely, in the limit $w \rightarrow 1$ it contains an exponentially behaved factor of $q^{-1/4}$. As a result of this the exponential behavior of the integrand near $w = 1$ has additional important contributions. In fact, it is easily seen that the principal behavior

of the integrand near $w=1$ is given by $q^{-1/3-u/4}$, where u is the channel with the double pole in the example of Fig. 5. More generally, for a diagram of this class the lines are always composed of two well-defined sets corresponding to the two windows of the dual surface.¹⁵ Or, put in simpler language, when drawn with quark lines there is one channel that has no quark lines running across it; i.e., the diagram can be pulled apart without breaking any quark lines. This serves to single out a particular channel called u . This channel is naturally identified as carrying vacuum quantum numbers only.

We have seen that the integrand has leading behavior near $w \rightarrow 1$ of the form $q^{-1/3-u/4}$. Expanding out the integrand in powers of q , it also contains less singular terms $q^{-1/3-u/4+n}$, $n=1, 2, \dots$. It therefore follows that any diagram of this class is well-defined (convergent) for $u < -\frac{4}{3}$, and contains singularities for

$$u = -\frac{4}{3} + 4n, \quad n=0, 1, 2, \dots \quad (5.6)$$

These singularities turn out to be logarithmic branch points. It is remarkable that their position is independent of α_0 . If one naively attempts to construct a counterterm in the same way as in Sec. IV (as one probably should) one would subtract a term with ψ replaced by $\bar{\psi}$ and ψ_T replaced by

$$\bar{\psi}_T = -[(\ln w)/2\pi]q^{-1/4}.$$

This would remove the branch point at $u = -\frac{4}{3}$, but all the higher ones would still survive.

The branch points we have just found are quite peculiar. They do not correspond to normal thresholds and they only occur in vacuum quantum numbers. Furthermore, they have definite angular momentum properties, lying, in fact, on a new kind of Regge cut. Since the branch point at $u = -\frac{4}{3} + 4n$ contains angular momentum $n, n-2, \dots, 0$ (for n even), it would seem that the J -plane cut trajectory is

$$\alpha_{\text{cut}}(u) = \frac{1}{3} + \frac{1}{4}u. \quad (5.7)$$

In fact, this result is verified by analyzing the asymptotic behavior of these diagrams. Notice that the slope of this Regge cut (just as in the case for the more

familiar ones in the crossed channels) is half of the slope of the input pole trajectories of the mode. There is a strong temptation to call these cuts the Pomeranchukon and be very happy. However, at least to the present order in g^2 , these singularities are unacceptable, since they apparently lead to a violation of unitarity. One might hope to find a procedure that eliminates these unwanted singularities from the physical sheet, perhaps at the same time introducing still another J -plane singularity related to the Pomeranchukon. Alternatively, the problem may be resolved when higher-order loops are included. We consider this to be an extremely important problem.

We should finally remark that cutting out the singular portion of the integration with a factor $\theta(1-w-\epsilon)$ removes the new branch points without altering the other absorptive parts. This kind of cutoff, however, in general leads to an unacceptable asymptotic behavior for the amplitudes. Nevertheless, as an interim measure, such a cutoff should be implicitly understood to be made for this class of diagrams in the discussions of the next section.

VI. ONE-LOOP UNITARITY

In this section we wish to make a modest beginning on the important question of unitarity. Clearly all the labors of the previous sections are of interest only if one can eventually enumerate an infinite set of diagrams whose sum is unitary. It should be emphasized that *a priori* it is not at all evident that any such selection of dual Feynman graphs can be made. Indeed, it could conceivably turn out that any set of dual Feynman diagrams, having the correct absorptive parts in any particular chosen channel, is not crossing symmetric (i.e., unitary in crossed channels as well). We shall partially answer this question by displaying a selection of N -point one-loop diagrams that is both unitary (in a perturbative sense) and crossing symmetric. For simplicity we only deal with connected diagrams. The connectedness structure of the S matrix is a standard problem, relevant for unitarity, but probably not the source of any serious difficulties.

The counting of dual Feynman diagrams with specific absorptive parts can be a rather tricky business because of their tendency to contain cuts in channels that are not manifest when the diagram is drawn in a particular way. For this reason, it turns out to be very useful, as an intermediate step in the discussion, to deal with "maximal-loop diagrams restricted to cubes." By "maximal-loop" we mean that the diagrams are drawn as one large loop with the external scalar states directly attached, as in Fig. 1. "Restriction to cubes" means that when the integration variables are introduced in the natural way described in Sec. II, the integrals are restricted to the region defined by the unit cube of these variables. This is just the sort of term we obtained by decomposing diagrams in Secs.

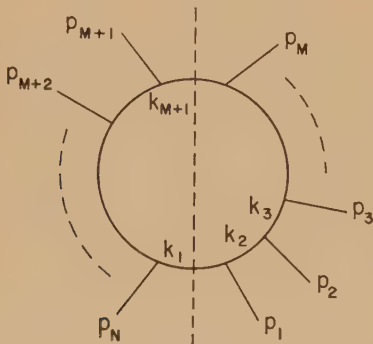


FIG. 6. $(M+N)$ -point loop diagram whose absorptive part is examined in the indicated channel.

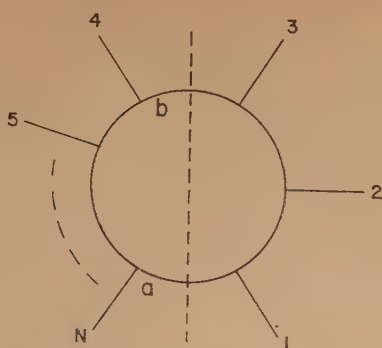


FIG. 7. Absorptive part of an N -point loop diagram in a three-body channel.

IV and V. It is important to remember that when an integration is restricted in this way, a twisted line may not be dualized.

We now state the rule for obtaining a unitary expression: For a particular ordering of the N external momenta, sum the 2^N terms obtained by allowing each of the N internal lines to be either twisted or untwisted. Then sum all inequivalent permutations of the external momenta. This gives a total of $2^{N-1}(N-1)!$ terms, each of which is a maximal-loop integral restricted to a cube. With this rule at hand we proceed to indicate why, aside from the spurious singularities found in Sec. V, the only absorptive parts of this sum of terms are the usual two-particle thresholds, and why the corresponding discontinuities are the appropriate products of $M+2$ and $N-M+2$ particle trees (see Fig. 6) integrated over two-body phase space.

The advantage of integrals restricted to cubes is that they only contain cuts in channels that are manifest as drawn. Therefore, to consider the absorptive part in a particular channel we may consider the subset of diagrams having the lines appropriately divided into two sets. Thus the M lines on the right in Fig. 6 may be arbitrarily permuted and similarly for the $N-M$ lines on the left. For a particular assignment of twists, we recall (in rather schematic form) the formula of Sec. II,

$$\int d^4k \int_0^1 \prod_{i=1}^N [du_i u_i^{-\alpha(k_i^2)-1}] \\ \times (1 \pm w)^2 [G(u_1, u_2, \dots, u_N)]^{\alpha_0-1} \\ \times \text{Tr} \{ V(\pm u_1)^R V(\pm u_2)^R \dots V(\pm u_N)^R \},$$

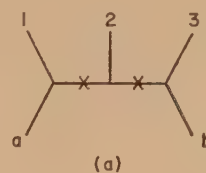
where the signs and G are chosen in accordance with the rules of Sec. II. To investigate the discontinuity across the two-body cut for which internal line No. 1 is in the state $|s\rangle$ and internal line No. $M+1$ is in the state $|s'\rangle$, one simply inserts $|s\rangle\langle s|$ and $|s'\rangle\langle s'|$ into the appropriate locations of the trace. Also, if these two states have $R|s\rangle = r|s\rangle$ and $R|s'\rangle = r'|s'\rangle$, one replaces $u_1^{R-\alpha(k_1^2)-1}$ by $\delta(r-\alpha(k_1^2))$, putting $u_1=0$ elsewhere and dropping the u_1 integration. Similarly, $u_{M+1}^{R-\alpha(k_{M+1}^2)-1}$ is replaced by $\delta(r'-\alpha(k_{M+1}^2))$. This rule is just the

natural analog of the Cutkosky prescription. It is somewhat oversimplified as stated because $|s\rangle$ and $|s'\rangle$ need not be eigenstates of R (whenever $\alpha_0 \neq 1$), but the generalization can be easily made. After carrying out the indicated substitutions, one is left with a two-dimensional phase-space integration (of the usual type) of the product of two tree diagrams, just as desired.

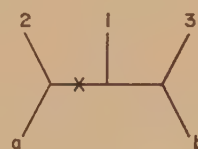
To complete the demonstration that the two-body discontinuity is correct, we should show that the sum of all the terms contributing on either side of a particular discontinuity is, in fact, the appropriate N -point function. The delicate point is that when any line is twisted, the corresponding factor in the discontinuity is *not* an N -point tree. Rather, because of the way in which the integration regions have been restricted, it is only part of a tree. Remarkably, all the terms combine to give precisely the desired sum of trees. Rather than proving this fact in all generality, we demonstrate it for a specific example.

Consider a discontinuity of the type shown in Fig. 7. For simplicity we shall take a and b to be ground states. Then, when all terms are summed, we expect to get as one factor the sum of all five-point trees made from particles $a, b, 1, 2$, and 3 . There are 12 such trees. To obtain these 12 trees we must combine the 24 terms arising from the six permutations of particles $1, 2$, and 3 and the four possible twist combinations for the two internal lines between them. The six terms with no twists are six of the desired trees. The remaining six trees are obtained by combining the other 18 terms in appropriate fashion.

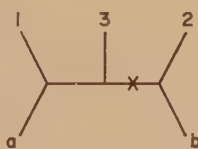
To see how the remaining terms are combined, consider the diagrams shown in Fig. 8. Any one of



(a)



(b)



(c)

FIG. 8. Three equivalent five-point functions.

these terms would represent the same tree if the integration limits were appropriate. However, the integration limits emerging from factorizing terms restricted to cubes are only portions of tree diagrams. In fact, the three terms in Fig. 8 combine to give the complete tree corresponding to any one of them. In this way, the 18 terms with twists give the six remaining trees. Let us sketch how the algebra works. Corresponding to Fig. 8(a), we obtain

$$\begin{aligned} F_a &= g^3 \int_0^1 dx_1 dx_2 x_1^{-\alpha_{a1}-1} x_2^{-\alpha_{3b}-1} (1+x_1)^{-p_1 \cdot p_2} \\ &\quad \times (1+x_2)^{-p_2 \cdot p_3} (1-x_1 x_2)^{\alpha_0-1-p_1 \cdot p_3} \\ &= g^3 \int_0^1 dx_1 dx_2 I(x_1, x_2). \end{aligned}$$

On the other hand, Fig. 6(b) gives

$$\begin{aligned} F_b &= g^3 \int_0^1 dy_1 dy_2 y_1^{-\alpha_{a2}-1} y_2^{-\alpha_{3b}-1} (1+y_1)^{-p_1 \cdot p_2} \\ &\quad \times (1-y_2)^{-\alpha_{13}-1} (1+y_1 y_2)^{-p_2 \cdot p_3}. \end{aligned}$$

The change of variables $y_1 = 1/x_1$, $y_2 = x_1 x_2$ brings F_b to the form

$$F_b = \int_{x_1 > 1, x_1 x_2 < 1} I(x_1, x_2) dx_1 dx_2.$$

In similar fashion, the diagram in Fig. 8(c) contributes

$$F_c = \int_{x_2 > 1, x_1 x_2 < 1} I(x_1, x_2) dx_1 dx_2.$$

Thus the three terms combine to give the complete five-point function

$$F = \int_{\Pi} dx_1 dx_2 \theta(1-x_1 x_2) I(x_1, x_2).$$

(A change of variables is still required to obtain the Bardakci-Ruegg form.) Not surprisingly, the corresponding algebra can be carried through for the general case.

As an additional comment, we point out that there are actually four terms of each type, which differ only in the twistedness of lines a and b , and thus superficially appear to contribute equally to the absorptive parts. Two of these terms are orientable and two are non-orientable. It is important to include all four terms in order that the intermediate states producing threshold branch points all have physical signature.

We have now shown that a certain manifestly crossing-symmetric sum of terms comprises the complete one-loop contribution to the connected part of the unitary S matrix. It is important for several reasons to reassemble these diagrams as dual Feynman diagrams

with the complete integration regions instead of the cube restrictions. First of all, this will lead to a simpler counting of diagrams and allow the dualization of any line, twisted or not. Secondly, the elimination of spurious states by means of projections in Sec. II was carried out for complete diagrams. When the diagrams are decomposed into cubes the individual terms will have absorptive parts corresponding to spurious intermediate states. If these spurious-particle thresholds are to be absent from the complete answer, it is important that the sum of diagrams can be expressed as a sum of complete duality diagrams without any pieces left over.

The identification of complete dual Feynman diagrams from the $2^{N-1}(N-1)!$ cube-restricted diagrams is actually quite easy using the rules of Secs. IV and V. The $(N-1)!/2$ terms without twists give directly the $(N-1)!/2$ inequivalent planar terms. These are related by duality transformations to the diagrams obtained by attaching the tadpole of Fig. 1(a) to each of the possible N -point tree diagrams. The $2^{N-2}(N-1)!$ terms with an odd number of twists are readily combined to give the $(N-1)!/2$ inequivalent nonorientable terms. These are related by duality transformations to the diagrams obtained by attaching the tadpole of Fig. 1(b) to each of the N -point tree diagrams. Finally, there are the terms with an even number of twists. These may also be combined to give complete duality diagrams. They may be related to the diagrams obtained by inserting the self-energy of Fig. 1(c) onto some line (external or internal) of a tree. If this insertion is made so as to divide the external lines into a set of M lines (containing a particular one) and $N-M$ lines, then, for $N > 4$, there are $(N-1)!/2(N-M)$ inequivalent diagrams of this type. This completes the classification of one-loop diagrams. We have found that a unitary answer requires adding each inequivalent dual Feynman diagram with equal weight, a remarkably simple result.

VII. CONCLUSIONS

We have given the rules for the explicit calculation of all one-loop dual Feynman diagrams with any number of twisted lines. These are expressed as integrals involving elliptic functions. The planar and the nonorientable diagrams were renormalized. We showed that the nonplanar orientable diagrams develop new singularities which are perhaps associated with the Pomeranchuk trajectory. Finally, we showed that the sum of all inequivalent dual Feynman diagrams, taken with equal weights, is unitary to this order of the perturbation series.

Let us now review the current stage of development of the theory and the problems that remain to be solved before a dual unitary theory can be written down. The analysis of the structure of dual Feynman graphs that we have given in Ref. 11 showed that all dual graphs could be constructed with the aid of the operators

corresponding to the four one-particle irreducible graphs of the theory (see Fig. 1). Three of these operators [Figs. 1(a)–1(c)] are one-loop operators and can therefore be constructed explicitly by factorizing the expressions we have given for the N -particle one-loop diagrams into a product of trees and the appropriate one-loop operator.¹⁷ In Ref. 10 we explicitly calculated the renormalized planar self-energy operator from a knowledge of the planar one-loop diagrams and their renormalization. It is not difficult to extract the planar tadpole operator [Fig. 1(a)], which can then be used in conjunction with the general vertex operator to construct all multiloop planar diagrams. (The self-energy operator can also be used directly.) The formulas of this paper allow one to construct, by techniques completely analogous to those of Ref. 10, the operators corresponding to Figs. 1(b) and 1(c) as well. In each case a reasonable counterterm operator is obtained by replacing the various ψ functions by the corresponding $\tilde{\psi}$ functions. Whether this prescription is valid for the self-energy operator of Fig. 1(c) remains to be seen.

The remaining problems in the implementation of the unitarization program are the following.

(1) One must deal with the new singularities generated by the nonplanar orientable graphs and perhaps relate them to the Pomeranchukon.

(2) The operator corresponding to Fig. 1(d) still needs to be constructed. Although we can write the operator expression for this graph, its explicit evaluation is quite difficult. Note that its renormalization is related to the renormalization of the nonplanar orientable graph because it contains this graph as a factor when two internal lines are cut.

(3) One must study the new divergences that appear when any of the four primitive renormalized operators is iterated.

(4) Consistency conditions and other criteria that may bear on the uniqueness of the procedure used to construct counterterms require further investigation.

(5) Once the above problems have been resolved, any given dual Feynman diagram can be constructed by iteration of the primitive operators. It then remains to determine the relative weights of the various terms in the perturbation series such that the resulting sum is perturbatively unitary. It is possible that one could then exhibit explicit integral equations which sum the complete set of diagrams. We have found that each inequivalent one-loop graph should be included with equal (unit) weight. It is perhaps reasonable to conjecture that a similar rule is also valid for multiloop graphs.

ACKNOWLEDGMENT

We wish to acknowledge gratefully the participation of Dr. T. H. Burnett in the early stages of this work.

¹⁷ The details of this construction and the forms of the primitive operators will be given elsewhere.

APPENDIX

In the text we were led to introduce the following four functions:

$$\psi(x) = -2\pi i \exp\left(\frac{\ln^2 x}{2 \ln w}\right) \times \theta_1\left(\frac{\ln x}{2\pi i} \middle| \frac{\ln w}{2\pi i}\right) / \theta_1'\left(0 \middle| \frac{\ln w}{2\pi i}\right), \quad (\text{A1})$$

$$\psi_N(x) = -2\pi i \exp\left(\frac{\ln^2 x}{2 \ln w}\right) \times \theta_1\left(\frac{\ln x}{2\pi i} \middle| \frac{\ln w}{2\pi i} + \frac{1}{2}\right) / \theta_1'\left(0 \middle| \frac{\ln w}{2\pi i} + \frac{1}{2}\right), \quad (\text{A2})$$

$$\psi_{NT}(x) = -2\pi i \exp\left(\frac{\ln^2 x}{2 \ln w}\right) \times \theta_1\left(\frac{\ln x}{2\pi i} + \frac{1}{2} \middle| \frac{\ln w}{2\pi i} + \frac{1}{2}\right) / \theta_1'\left(0 \middle| \frac{\ln w}{2\pi i} + \frac{1}{2}\right), \quad (\text{A3})$$

$$\psi_T(x) = 2\pi \exp\left(\frac{\ln^2 x}{2 \ln w}\right) \times \theta_1\left(\frac{\ln x}{2\pi i} + \frac{1}{2} \middle| \frac{\ln w}{2\pi i}\right) / \theta_1'\left(0 \middle| \frac{\ln w}{2\pi i}\right). \quad (\text{A4})$$

$\theta_1(\nu | \tau)$ is a standard Jacobi θ function in the notation of the Bateman project. In each case we require an accurate asymptotic estimate of the function for $w \rightarrow 1$. For this purpose one requires the transformation formula

$$\theta_1\left(\frac{\nu}{\alpha + \beta\tau} \middle| \frac{\gamma + \delta\tau}{\alpha + \beta\tau}\right) = \epsilon(\alpha + \beta\tau)^{1/2} \times \exp\left(\frac{i\pi\beta\nu^2}{\alpha + \beta\tau}\right) \theta_1(\nu | \tau), \quad (\text{A5})$$

where $\epsilon^8 = 1$ and α, β, γ , and δ are integers satisfying $\alpha\delta - \beta\gamma = 1$.

For the first function, $\psi(x)$, we choose $\alpha = \delta = 0$ and $\gamma = -\beta = 1$ to show that

$$\begin{aligned} \psi(x) &= -\ln w \theta_1\left(\frac{\ln x}{\ln w} \middle| \frac{-2\pi i}{\ln w}\right) / \theta_1'\left(0 \middle| \frac{-2\pi i}{\ln w}\right) \\ &= -\frac{\ln w}{\pi} \sin\left(\frac{\pi \ln x}{\ln w}\right) \\ &\quad \times \prod_{n=1}^{\infty} \left[\frac{1 - 2q^{2n} \cos(2\pi \ln x / \ln w) + q^{4n}}{(1 - q^{2n})^2} \right], \quad (\text{A6}) \end{aligned}$$

where

$$q = e^{2\pi^2 / \ln w}.$$

This formula provides an extremely good estimate of $\psi(x)$ near $w=1$. We see that $\psi(x)$ differs from

$$\tilde{\psi}(x) = \frac{-\ln w}{\pi} \sin\left(\frac{\pi \ln x}{\ln w}\right) \quad (\text{A7})$$

only by $O(q^2)$. The identity

$$\psi(x) = \psi(w/x) \quad (\text{A8})$$

is also manifest in (A6).

For the second function, $\psi_N(x)$, the choice $\beta=-2$, $\gamma=0$, $\alpha=\delta=1$ in (A5) and some simple manipulations allow one to show

$$\begin{aligned} \psi_N(x) &= -2 \ln w \theta_1\left(\frac{\ln x}{2 \ln w} \middle| -\frac{1}{2} - \frac{\pi i}{2 \ln w}\right) / \\ &\quad \theta_1'\left(0 \middle| -\frac{1}{2} - \frac{\pi i}{2 \ln w}\right) \\ &= -\frac{2 \ln w}{\pi} \sin\left(\frac{\pi \ln x}{2 \ln w}\right) \\ &\quad \times \prod_{n=1}^{\infty} \left[\frac{1 - 2q_N^{2n} \cos(\pi \ln x / \ln w) + q_N^{4n}}{(1 - q_N^{2n})^2} \right], \quad (\text{A9}) \end{aligned}$$

where

$$q_N = -ie^{\pi^2/2 \ln w}.$$

From Eq. (A9) it is clear that

$$\tilde{\psi}_N(x) = -\frac{2 \ln w}{\pi} \sin\left(\frac{\pi \ln x}{2 \ln w}\right) \quad (\text{A10})$$

differs from $\psi_N(x)$ by $O(q_N^2)$ as $w \rightarrow 1$. The symmetry

$$\psi_N(x) = -\psi_N(1/x) \quad (\text{A11})$$

is also evident.

For the third function, $\psi_{NT}(x)$, one uses the same transformation formula as for $\psi_N(x)$, together with additional identities

$$\theta_1(\nu | \tau) = -e^{-i\tau(2\nu-\tau)} \theta_1(\nu-\tau | \tau) \quad (\text{A12})$$

and

$$\theta_1\left(\frac{1}{2} - \nu | \tau\right) = \theta_2(\nu | \tau) \quad (\text{A13})$$

to show that

$$\begin{aligned} \psi_{NT}(x) &= 2 \ln w \theta_2\left(\frac{\ln x}{2 \ln w} \middle| -\frac{1}{2} - \frac{\pi i}{2 \ln w}\right) / \\ &\quad \theta_1'\left(0 \middle| -\frac{1}{2} - \frac{\pi i}{2 \ln w}\right) \\ &= -\frac{2 \ln w}{\pi} \cos\left(\frac{\pi \ln x}{2 \ln w}\right) \\ &\quad \times \prod_{n=1}^{\infty} \left[\frac{1 + 2q_N^{2n} \cos(\pi \ln x / \ln w) + q_N^{4n}}{(1 - q_N^{2n})^2} \right]. \quad (\text{A14}) \end{aligned}$$

Comparing (A9) and (A14) gives the additional symmetries

$$\psi_{NT}(x) = \psi_N(w/x), \quad \psi_{NT}(x) = \psi_{NT}(1/x). \quad (\text{A15})$$

Also, the function

$$\tilde{\psi}_{NT}(x) = -2 \frac{\ln w}{\pi} \cos\left(\frac{\pi \ln x}{2 \ln w}\right) \quad (\text{A16})$$

agrees with $\psi_N(x)$ to $O(q_N^2)$ for $w \rightarrow 1$.

Finally, for $\psi_T(x)$, we use the same transformation formula as for $\psi(x)$ together with the identity

$$\theta_1(\nu + \frac{1}{2} \tau | \tau) = ie^{-i\pi(\nu+\tau/4)} \theta_4(\nu | \tau) \quad (\text{A17})$$

to deduce that

$$\begin{aligned} \psi_T(x) &= -\ln w \theta_4\left(\frac{\ln x}{\ln w} \middle| -\frac{2\pi i}{\ln w}\right) / \theta_1'\left(0 \middle| -\frac{2\pi i}{\ln w}\right) \\ &= -\frac{1}{2} \ln w q^{-1/4} \\ &\quad \times \prod_{n=1}^{\infty} \left[\frac{1 - 2q^{2n-1} \cos(2\pi \ln x / \ln w) + q^{4n-2}}{(1 - q^{2n})^2} \right]. \quad (\text{A18}) \end{aligned}$$

This function has the manifest symmetries

$$\psi_T(x) = \psi_T(1/x) = \psi_T(w/x). \quad (\text{A19})$$

Also, it contains the exponential factor $q^{-1/4}$ and differs from

$$\tilde{\psi}_T(x) = -[(\ln w)/2\pi] q^{-1/4} \quad (\text{A20})$$

by $O(q)$ as $w \rightarrow 1$.

RENORMALIZATION IN THE DUAL RESONANCE MODEL. ITS ARBITRARINESS IN THE GENERAL CASE AND FOR UNIT INTERCEPT

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Abstract: It is shown in the case of arbitrary intercept that the finite part of the planar one-loop diagram can be defined by imposing duality, Regge behaviour with linear trajectories and unitarity, up to a class of functions of the external momenta depending on an infinite number of arbitrary parameters.

In the Virasoro case of unit intercept, it turns out that the infinite parts of the planar and non-orientable one-loop diagrams factorize into a finite momentum dependent part and two infinite constants. Always in that case it is argued that the renormalization of all planar graphs depends on a single constant.

1. INTRODUCTION

The implementation of the unitarization program of the dual resonance models finds its justification not only in the construction of a dual, Regge behaved, unitary model of the hadrons dynamics having nice mathematical features, but also in the hope that the perturbation expansion thereby defined could be used, possibly in lower orders in the coupling constant, to compute physical quantities, like the masses of resonances, widths, and so on.

Several problems are still to be solved before the theory can give physical values. The first is certainly the inclusion of spin and internal symmetries in the model. The second problem is to define a finite, Regge-behaved and perturbative unitary expansion of the amplitude in the model with scalar external particles, and to determine on how many arbitrary constants this expansion depends. This is what we call the renormalization program. The third problem is to extract from this renormalized expansion the physical spectrum and amplitude. This problem is not at all trivial since the counting of the graphs in a dual theory is not the same as in a Lagrangian theory and this makes the interpretation of a unitarized propaga-

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tor rather delicate; one easily sees that in a unitarized five-point function for instance, the unitarized propagator should be used only once and not twice, otherwise the unitary counting would be wrong already at order g^2 .

In this paper we concern ourselves only with the problem of defining a renormalized perturbation expansion and to determine the degree of its arbitrariness, within the framework of simple rules: unitarity, duality, Regge behaviour. Therefore we examine the work already done on the renormalization of the one-loop planar graph [1-3]. It is found that the integrand of the counter-term, which is the infinite part of the diagram is defined only up to a function of one variable, therefore up to an infinite number of parameters.

This holds for the case of general intercept; however it is not obvious that in this case the theory is unrenormalizable since other approaches like eventual dispersion relations might restrict the arbitrariness.

The Virasoro case of unit intercept [4] is very interesting for the renormalization program since it turns out that the loop diagram is the sum of a finite function and of a function depending multiplicatively on an infinite constant. This can be generalized also to the tadpole vector [5] which can be written as the sum of a finite vector which we call the renormalized tadpole vector and of an infinite vector which we call the counter-term tadpole vector. The counter term is infinite because it depends multiplicatively on an infinite constant which we call C_2 . As an infinite constant resulting from a divergent integral is defined always up to a finite constant it turns out that both renormalized tadpole and counter term depend on one arbitrary parameter, which is the arbitrary finite part of the constant C_2 . A similar result is true also for the non-orientable diagram, and the counter terms of the planar and the non-orientable graphs are simply proportional.

At higher order in the coupling constant g , one has to introduce new counter terms to cancel additional divergences [6, 7]. At order g^3 and in the Virasoro case, it turns out that the addition of these new counter terms simply changes the constant C_2 into a new constant $C_2 + g^2 C_4$. Therefore one can reasonably expect that at any order g^{2N-1} it is possible to define a multiplicative constant $C_2 + \dots + g^{2(N-1)} C_{2N}$ such that the iteration, up to order g^{2N-1} of the renormalized tadpole operator, gives finite result.

Therefore it is possible in the case of unit intercept to define a finite, perturbative unitary and dual expansion of the planar tadpole vector depending on only one constant. However this result has not yet found a simple interpretation in terms of usual quantum-field-theory renormalization because of the counting problem.

2. PROPERTIES OF THE ONE-LOOP PLANAR GRAPH

The expression of the one-loop planar graph, as computed in refs. [3, 5] is

$$\begin{aligned}
 & F_1^{(N)}(p_1, \dots, p_N) \\
 &= 4\pi^2 g^N \int_0^1 \prod_{i=1}^N \{dx_i (1-x)^{\alpha_0-1}\} w^{-\alpha_0-1} [f(w)]^{-4} (\ln w)^{-2} (1-w) \\
 &\quad \times \prod_{1 \leq i < j \leq N} \Psi\left(\frac{\rho_j}{\rho_i}\right)^{-p_i \cdot p_j}, \quad (2.1)
 \end{aligned}$$

where $\rho_i = x_1 x_2 \dots x_i$, $f(w) = \prod_{n=1}^{\infty} (1-w^n)$ and $w = x_1 \dots x_N$. The function $\Psi(x, w)$ is equal to

$$\Psi(x, w) = \exp - \frac{\ln(x) \ln(wx^{-1})}{2 \ln w} \prod_{n=1}^{\infty} \left\{ \frac{(1-w^{n-1}x)(1-w^n x^{-1})}{(1-w^n)^2} \right\} \quad (2.2)$$

and using the theory of elliptic functions one shows that

$$\Psi(x, w) = - \frac{\ln w}{\pi} \sin \pi \frac{\ln x}{\ln w} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n} \cos(2\pi \ln x / \ln w) + q^{4n})}{(1 - q^{2n})^2}, \quad (2.3)$$

where $q = \exp 2\pi^2 / \ln w$. As a result, when $w \rightarrow 1$

$$\Psi(x, w) = - \frac{\ln w}{\pi} \sin \pi \frac{\ln x}{\ln w} + O(q^2), \quad (2.4)$$

$\Psi(x, w)$ satisfies the following properties:

$$\Psi(x, w) = \Psi\left(\frac{w}{x}, w\right), \quad (2.5)$$

$$\Psi(x, w) = - \Psi\left(\frac{1}{x}, w\right), \quad (2.6)$$

$$\lim_{x \rightarrow 1} (1-x)^{-1} \Psi(x, w) = 1. \quad (2.7)$$

From duality, we should expect $F_1^{(N)}$ to satisfy the identities of fig. 1. This means that there must exist a self-energy operator $\Sigma(p)$ and a tadpole vector T such that:

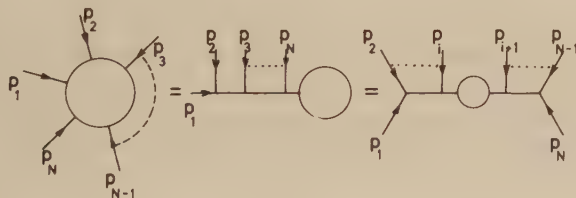


Fig. 1. Duality identities satisfied by the one-loop graph.

$$\langle 0 | VD \dots D \Sigma D \dots V | 0 \rangle = TDV \dots V | 0 \rangle = F_1^{(N)} . \quad (2.8)$$

As it stands this equality is valid between multiperipheral trees, but the operator Σ and the vector T can be applied on any kind of tree and reproduce $F_1^{(N)}$, provided that projected operators are used.

These operators have been constructed explicitly (refs. [3, 5]). For instance, T has the following form:

$$T = 4\pi^2 g \int_0^1 dw w^{-\alpha_0-1} [f(w)]^{-4} (\ln w)^{-2} \langle 0 | \exp \{ (a | (1-w)E(1-w) | a) \} , \quad (2.9)$$

where

$$(a | X | a) = \sum_{m,n=1}^{\infty} a_{\mu}^{(m)}(X)_{mn} a_{\mu}^{(n)} ,$$

$$(1-w)_{mn} = (1-w)^m \delta_{mn}$$

and

$$(E)_{mn} = \frac{1}{2} \frac{\sqrt{mn}}{m!n!} \left(-\frac{\partial}{\partial x} \right)^m \left(-\frac{\partial}{\partial y} \right)^n \ln \left[\frac{y-x}{\Psi(x/y)} \right]_{x=y=1} . \quad (2.10)$$

A remarkable property of T is that if we compute $TDV \dots V | 0 \rangle$ to show that this is equal to $F_1^{(N)}$ we need only to make a change of variables and use the properties of eqs. (2.5)-(2.7) of ψ , but never have to write what ψ is.

This is due to the fact that the evaluation of E between coherent states only needs computation of matrix elements of the form

$$(p_i | (1-a_i)E(1-b_j) | p_j) \quad \text{where} \quad |p_i\rangle = \prod_{n=1}^{\infty} \left| \frac{p_i}{\sqrt{n}} \right\rangle$$

and using eq. (2.10) one sees easily that this expression is nothing else than a Taylor expression. Using eqs. (2.5)-(2.7) it is found to be equal to

$$\frac{1}{2} p_i \cdot p_j \ln \left[\frac{a_i - b_j}{\psi(a_i/b_j)} \frac{\psi(a_i) \psi(b_j)}{1 - a_i \quad 1 - b_j} \right] . \quad (2.11)$$

Such a property holds also true for the operator $\Sigma(p)$ which can be also considered as a functional of ψ , $\Sigma(p, \psi)$ which, when inserted between trees reproduces the expression (2.1), regardless of what ψ is, provided that it satisfies eqs. (2.5)-(2.7).

It is this crucial property which will enable us to construct a dual factorisable counter term.

3. RENORMALIZATION OF THE ONE-LOOP DIAGRAM AND ITS ARBITRARINESS

We are looking for a function $\tilde{F}_1^{(n)}(p_1, \dots, p_N)$ which we call counter-term, satisfying the following conditions:

- (1) $F_1^{(N)} - \tilde{F}_1^{(N)}$ is finite*.
- (2) $\tilde{F}_1^{(N)}$ has the same duality properties as $F_1^{(N)}$ (see fig. 2) and must be factorisable within the same space of harmonic oscillators as $F_1^{(N)}$.
- (3) $\tilde{F}_1^{(N)}$ can be expanded as a sum of single or double poles of a given channel, with no entire function of the trajectory $\alpha(p^2)$ of the channel added. The poles of $\tilde{F}_1^{(N)}$ must be on the same trajectories as those of the N -point function. In this way we can ensure Regge behaviour of $\tilde{F}_1^{(N)}$ and also the unitarity of $F_1^{(N)} - \tilde{F}_1^{(N)}^\dagger$.

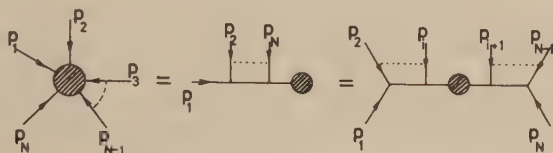


Fig. 2. Duality identities satisfied by the one-loop counter term.

The conditions (1) and (2) are obviously satisfied when we chose

$$\begin{aligned} \tilde{F}_1^{(N)} = 4\pi^2 g^N \int_0^1 \prod_{i=1}^N \{dx_i (1-x_i)\}^{\alpha_0-1} w^{-\alpha_0-1} [f(w)]^{-4} (1-w)(\ln w)^{-2} G(w) \\ \times \prod_{1 \leq i < j \leq N} \tilde{\Psi} \left(\frac{\rho_j}{\rho_i} \right)^{-p_i \cdot p_j}, \quad (3.1) \end{aligned}$$

where

$$G(w) = 1 + O(q^2) \quad \text{when } w \rightarrow 1, \quad (3.2)$$

$$[\Psi - \tilde{\Psi}](x, w) \approx O(q^2) \quad \text{when } w \rightarrow 1. \quad (3.3)$$

* By 'finite' we mean in fact that the exponential divergence near $w = 1$ is removed.

However $F_1^{(N)} - \tilde{F}_1^{(N)}$ is still infinite because it can be rewritten as a tadpole put on an external on-mass-shell line, therefore creating a divergence. This divergence is removed when we consider operators like Σ or vector T . However, in the $\alpha_0 = 1$ case, infrared divergences also appear. We do not deal here with these still unresolved problems.

† The Regge behaviour in the complex s, t plane for $F_1^{(4)}$ itself can in fact be inferred from computer calculations. Therefore we expect $F_1^{(N)} - \tilde{F}_1^{(N)}$ to be both finite and Regge behaved.

Condition (1) is thus satisfied automatically. To satisfy (2) we impose that $\tilde{\Psi}$ satisfy properties (2.5)-(2.7). Then if in Σ and T we replace Ψ everywhere by $\tilde{\Psi}$, and multiply the integration measure by $G(w)$ we obtain an operator $\tilde{\Sigma}$ and a vector $\tilde{T}$ which reproduce $\tilde{F}_1^{(N)}$ when used on tree in exactly the same way as Σ and T reproduce $F_1^{(N)}$.

Of course we cannot prove that this form is the most general form satisfying (1) and (2). However any generalisation of this like replacing $G(w)$ by $G(w, x_i)$ gives rise to very hard problems (in this case, factorisation of satellites*) and is very unlikely to work. Let us impose condition (3) on $\tilde{F}_1^{(N)}$. This condition is extremely stringent. Let us look at the duality relation of fig. 3.

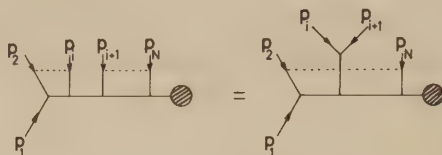


Fig. 3. Duality in a given channel for the counter term.

This condition is ensured for the loop diagram by construction as well as for the counter term. However it must be taken in a deeper sense for the counter term than for the loop. For the loop, we cannot expand the amplitude as a sum of poles in the channel $\alpha(p_i + p_{i+1})^2$ because there is a cut in the channel, even if the diagram does not exhibit the cut. However, for the counter term condition (3) tells us to take the diagram quite seriously, imposing that we may expand $\tilde{F}_1^{(N)}$ as a sum of poles in the channel $\alpha(p_i + p_{i+1})^2$. Thus $\tilde{F}_1^{(N)}$ must have an expression very close to the one of the original Born term.

A way to ensure (3) is to choose $\tilde{\Psi}$ as the Neumann function of the disc without hole:

$$\tilde{\Psi}\left(\frac{\rho_j}{\rho_i}, w\right) = r(w) \sin[\theta(\rho_j) - \theta(\rho_i)] = \frac{1}{2}r(w) |Z_j - Z_i|, \quad Z_j = e^{2i\theta(\rho_j)}. \quad (3.4)$$

The factor $r(w)$ can be eliminated by momentum conservation. In this way, up to a change of variable the integrand part which depends on the external momenta is identically the same as the corresponding part of the N -point function.

We have no way to prove that this is the only choice which satisfies (3). However we believe that requiring the same spectrum for $\tilde{F}_1^{(N)}$ as the one of $B_{(N)}$ implies choosing the same integrand for the definition of $\tilde{F}_1^{(N)}$.

* See for instance ref. [9].

Imposing now the conditions

$$(1) \quad \theta(\rho_j) - \theta(\rho_i) = \theta(\rho_j/\rho_i), \quad (3.5)$$

$$(2) \quad \tilde{\Psi}(x, w) = \tilde{\Psi}\left(\frac{w}{x}, w\right) \quad \text{cyclic symmetry}, \quad (3.6)$$

$$(3) \quad \lim_{x \rightarrow 1} \frac{\tilde{\Psi}(x, w)}{1-x} = 1, \quad (3.7)$$

we get from (1) that $\theta(\rho_j) = \pi \ln \rho_j / \sigma(w)$, from (2) that $\sigma(w) = \ln w$, and from (3) we get $r(w) = -(\ln w)/\pi$. One can also check that property (2.6) is satisfied.

Further the function $\tilde{\Psi}$ thus defined:

$$\tilde{\Psi} = -\frac{\ln w}{\pi} \sin \pi \frac{\ln x}{\ln w}, \quad (3.8)$$

satisfies the requirement (3.3) due to equality (2.4).

The counter term we define in this way satisfies all the required properties. However it is defined up to an arbitrary function $G(w)$ and thus in general $\tilde{F}_1^{(N)}$ is defined up to some function of the p_i . As no requirement has been found to determine G , it seems that in the general case, there is an infinite degree of freedom to define the real part of $F_1^{(N)}$.

However in the case of unit intercept the arbitrariness can be recast on only one constant; this we show below.

We can thus define a canonical counter term as the one where $G(w) = 1$. Any counter term can thus be split as a sum of the canonical counter term and of a finite counter term where the factor $w^{-\alpha_0-1} f(w)^{-4} (1-w)/\ln^2 w$ has been replaced by a function $\gamma(w)$ such that the integration near $w = 1$ does not create a divergence

$$\tilde{F}_1^{(N)} = \tilde{F}_{1,c}^{(N)} + \tilde{F}_{1,F}^{(N)}.$$

The function $\tilde{F}_{1,c}^{(N)}$ is thus uniquely defined while $\tilde{F}_{1,F}^{(N)}$ is defined up to a function of w .

In the following we will speak of 'the counter term' as the sum of $\tilde{F}_{1,c}^{(N)}$ and $\tilde{F}_{1,F}^{(N)}$.

4. EXPRESSION OF $F_1^{(N)}$, $\tilde{F}_1^{(N)}$ IN THE VIRASORO CASE

In the Virasoro case there is an infinite number of Ward identities [4] and

an infinite number of spurious states to eliminate. It is also the only case where there are enough Ward identities to eliminate all the ghosts.

To eliminate spurious states from the loop one has first to construct a projection operator on the subspace of real states. Such a projection operator has been defined for on-mass-shell states ($\alpha(p^2) = N$) in a recursive way by Brower and Thorn [8]. They found that after elimination of spurious states the loop has the following expression:

$$F_1^{(N)}(p_1 \dots p_N) = 4\pi^2 g^N \int_0^1 \prod_{i=1}^N dx_i w^{-2} [f(w)]^{-3} (1-w)(\ln w)^{-2} \\ \times \prod_{1 \leq i < j \leq N} \Psi\left(\frac{\rho_j}{\rho_i}\right)^{-p_i p_j}. \quad (4.1)$$

Therefore we can define the counter term as follows:

$$\tilde{F}_1^{(N)}(p_1 \dots p_N) = 4\pi^2 g^N \int_0^1 \prod_{i=1}^N dx_i \{w^{-2} [f(w)]^{-3} (1-w)(\ln w)^{-2} + \gamma(w)\} \\ \times \prod_{1 \leq i < j \leq N} \tilde{\Psi}\left(\frac{\rho_j}{\rho_i}\right)^{-p_i p_j}. \quad (4.2)$$

We notice that in the expression (2.3) of Ψ , the angles

$$\theta_j = 2\pi \frac{\ln x_1 x_2 \dots x_j}{\ln w} \quad (4.3)$$

play a natural role. Therefore we choose $\theta_1, \theta_2, \dots, \theta_{N-1}$ and w as new variables. The angles θ_i are ordered on the unit circle:

$$0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_{N-1} \leq 2\pi. \quad (4.4)$$

The Jacobian is

$$dx_1 \dots dx_N = \left(-\frac{\ln w}{2\pi}\right)^{N-1} d\theta_1 \dots d\theta_{N-1} dw. \quad (4.5)$$

Taking out of expression (2.3) the factor $-\ln w/2\pi$ we get a contribution

$$\prod_{1 \leq i < j \leq N} \left(-\frac{\ln w}{2\pi}\right)^{-p_i p_j} = \left(-\frac{\ln w}{2\pi}\right)^{\frac{1}{2} N p^2} = \left(-\frac{\ln w}{2\pi}\right)^{-N\alpha_0} = \left(-\frac{\ln w}{2\pi}\right)^{-N}. \quad (4.6)$$

This factor and the Jacobian give the contribution

$$\left(-\frac{\ln w}{2\pi}\right)^{N(1-\alpha_0)-1} = -\frac{2\pi}{\ln w}. \quad (4.7)$$

Therefore we find after the change of variable

$$F_1^{(N)}(p_1 \dots p_N) = -8\pi^3 g^N \int \prod_{i=1}^{N-1} d\theta_i \int_0^1 dw w^{-2} [f(w)]^{-3} (1-w)(\ln w)^{-3} \\ \times \prod_{1 \leq i < j \leq N} \varphi(\theta_j - \theta_i; w)^{-p_i \cdot p_j}, \quad (4.8)$$

where

$$\varphi(\theta, w) = 2 \sin \frac{1}{2} \theta \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n} \cos 2\theta + q^{4n})}{(1 - q^{2n})^2}, \quad q = e^{\frac{2\pi^2}{\ln w}} \quad (4.9)$$

and

$$\tilde{F}_1^{(N)}(p_1 \dots p_N) = -8\pi^3 g^N \int \prod_{i=1}^{N-1} d\theta_i \int_0^1 dw \{w^{-2} f(w)^{-3} (1-w)(\ln w)^{-1}\} \\ \times \prod_{1 \leq i < j \leq N} \tilde{\varphi}(\theta_j - \theta_i)^{-p_i \cdot p_j}, \quad (4.10)$$

where

$$\varphi(\theta) = \varphi(\theta, 1) = 2 \sin \frac{1}{2} \theta. \quad (4.11)$$

Thus we see that the integrand of the counter term has the Koba-Nielsen form. Further we notice that as φ is independent of w in the counter term we may integrate first over w and write

$$F_1^{(N)}(p_1 \dots p_N) = g^2 C_2 \mathcal{F}_1^{(N)}(p_1 \dots p_N), \quad (4.12)$$

where

$$\mathcal{F}_1^{(N)}(p_1 \dots p_N) \\ = g^{N-2} \int \prod_{i=1}^{N-1} d\theta_i \prod_{1 \leq i < j \leq N} [2 \sin \frac{1}{2} (\theta_j - \theta_i)]^{-p_i \cdot p_j}, \quad (4.13) \\ 0 \leq \theta_1 \dots \leq \theta_{N-1} \leq 2\pi$$

$$C_2 = -8\pi^3 \int_0^1 dw h(w), \quad (4.14)$$

$$h(w) = w^{-2} f(w)^{-3} (1-w)(\ln w)^{-3} + \gamma(w)(\ln w)^{-1}. \quad (4.15)$$

We can rephrase this result by writing

$$F_1^{(N)} = (F_1^{(N)} - \tilde{F}_1^{(N)}) + \tilde{F}_1^{(N)}, \quad (4.16)$$

where $F_1^{(N)} - \tilde{F}_1^{(N)}$ is now a convergent integral near $w = 1$ and is called the finite part of $F_1^{(N)}$. The function $\tilde{F}_1^{(N)}$ is its infinite part. Eq. (4.12) tells us that the divergence of the infinite part is multiplicative.

The freedom of adding a finite counter term becomes the freedom of defining the finite part of the divergent integral C_2 . It is natural that such a freedom exists, since an infinite constant is always defined up to a finite one. We can also define C_2 properly if we always understand a cut-off $0 \leq w \leq 1 - \epsilon$.

So, we can conclude that in the case of unit intercept, and only in this case, the ratio of two counter terms is a constant, independent of the external momenta.

We show below that the multiplicative dependence of $\tilde{F}_1^{(N)}$ on the constant C_2 can also be expressed in the operator formalism. The counter term tadpole which reproduces the counter term when it is inserted on a tree is the following:

$$\tilde{T} = 4\pi^2 g \int_0^1 dw h(w) \ln w (1-w)^{-1} \langle 0 | \exp(a | (1-w)\tilde{E}(1-w) | a) \rangle, \quad (4.17)$$

$$(\tilde{E})_{mn} = \frac{1}{2} \frac{\sqrt{mn}}{m!n!} \left(-\frac{\partial}{\partial x}\right)^m \left(-\frac{\partial}{\partial y}\right)^n \ln \left[\frac{y-x}{\tilde{\Psi}(x/y)} \right]_{x=y=1}. \quad (4.18)$$

The quantity $\tilde{T}$ is such that

$$\tilde{T} | \text{tree} \rangle = \tilde{F}_1^{(N)} = g^2 C_2 \mathcal{F}_1^{(N)}(p_1, \dots, p_N). \quad (4.19)$$

Thus the operator $\tilde{t}(h)$ defined by:

$$\tilde{t}(h) = - \frac{g \int_0^1 dw h(w) \ln w (1-w)^{-1} \langle 0 | \exp(a | (1-w)\tilde{E}(1-w) | a) \rangle}{2\pi \int_0^1 dw h(w)} \quad (4.20)$$

and is such that

$$\tilde{t} | \text{tree} \rangle = g^2 \mathcal{F}_1^{(N)}(p_1 \dots p_N). \quad (4.21)$$

As the function $\mathcal{F}_1^{(N)}(p_1 \dots p_N)$ is independent of the arbitrary function $h(w)$, eq. (4.20) defines a class of vectors $\tilde{t}(h)$ depending on the function h such that all these vectors have the same scalar products with external multiperipheral trees.

So if we introduce the projection operator $P(p)$ we may write

$$\tilde{t}(h)P(0) = \tilde{t}(h')P(0), \quad (4.22)$$

where h, h' are two different functions of w .

A simple form of $\tilde{t}$ is obtained if we choose $\gamma(w)$ in such a way that $h(w) = \delta(w - w_0)$, where w_0 is any number between 0 and 1:

$$\tilde{t} = -\frac{g}{2\pi} \frac{\ln w_0}{(1-w_0)} \langle 0 | \exp(a(1-w_0)E(w_0)(1-w_0) | a) . \quad (4.23)$$

An even more simple form of t is obtained if we let now $w_0 \rightarrow 1$. Then:

$$[\tilde{E}(w_0)]_{mn} \approx [E(w_0)]_{mn}$$

$$= \frac{1}{2\sqrt{mn} \ln w_0} + \frac{1}{2} \sum_{p \neq 0} \left(\frac{1}{1-w_0} \right)^m (M_+)_{mn} \left(\frac{-w_0^p}{1-w_0^p} \right)^n .$$

Therefore

$$\lim_{w_0 \rightarrow 1} (a | (1-w_0)\tilde{E}(w_0)(1-w_0) | a) = \frac{1}{2} (a | \sum_{p \neq 0} \left(\frac{1}{p} \right) M_+ \left(\frac{-1}{p} \right) | a) , \quad (4.24)$$

$$\tilde{t} = \frac{g}{2\pi} \langle 0 | \exp \frac{1}{2} (a | \Delta | a) , \quad (4.25)$$

with

$$\tilde{\Delta} = \sum_{p \neq 0} \left(\frac{1}{p} \right) M_+ \left(\frac{-1}{p} \right) , \quad (4.26)$$

one can check directly under this last form that

$$\tilde{t} | \text{tree} \rangle = g^2 \mathcal{F}_1^{(N)}(p_1 \dots p_N) . \quad (4.27)$$

The key equation to check this is the following [4]:

$$\begin{aligned} (p_j \rho_j | \Delta | p_i \rho_i) &= -p_i p_j \sum_{p \neq 0} \ln \left\{ \frac{(1 - (\rho_j - \rho_i)p^{-1})}{(1 - \rho_j p^{-1})(1 - \rho_i p^{-1})} \right\} \\ &= p_i \cdot p_j \ln \left\{ \frac{\pi(\rho_j - \rho_i)}{\sin \pi(\rho_j - \rho_i)} \frac{\sin \pi \rho_i}{\pi \rho_i} \frac{\sin \pi \rho_j}{\pi \rho_j} \right\} . \end{aligned} \quad (4.28)$$

The tree vector has the following expression:

$$| \text{tree} \rangle = g^{N-1} \int \prod_{i=2}^N dx_i x_i^{-\alpha(\pi_i^2)-1} \prod_{2 \leq i < j \leq N} \left(1 - \frac{\rho_i}{\rho_j} \right)^{-p_i p_j} \left| \sum_{j=2}^N p_j \rho_j^n \right\rangle ,$$

$$\rho_j = x_j x_{j+1} \dots x_N , \quad j = 2, \dots, N , \quad \rho_1 = 0 , \quad \rho_{N+1} = 1 . \quad (4.29)$$

To check eq. (4.27) we need only to use eq. (4.28) and to make the change of variables:

$$\theta_i = 2\pi(1 - \rho_N + \rho_i), \quad i = 1, \dots, N-1. \quad (4.30)$$

As a conclusion the vector $\tilde{T}$ is such that:

$$\tilde{T}P(0) = g(2\pi)^{-1} C_2 \langle 0 | \exp \left\{ \frac{1}{2} \langle a | \tilde{\Delta} | a \rangle \right\} P(0) \quad (4.31)$$

and under this simple form, the multiplicative dependence of the vector T on the constant C_2 is made obvious. This explicit form of T could also eventually be used for computing multi-counter-term graphs.

5. NON-ORIENTABLE GRAPHS FOR UNIT INTERCEPT

The case of the non-orientable one-loop planar graph is very similar to the case of the planar one-loop graph because both can be generated by tadpoles and therefore the divergence does not depend on external momenta, as it is the case for the non-planar orientable graph.

When spurious states are removed, the expression of the one-loop non-orientable graph is the following:

$$F_{1,N}^{(M)}(p_1, \dots, p_M) = 4\pi^2 g^M \int \prod_{i=1}^M du_i (1+w)f(-w)^{-3} w^{-2} (\ln w)^{-2} \\ \times \prod_{1 \leq i < j \leq M} \Psi_N(C_{ji})^{-p_i \cdot p_j}, \quad (5.1)$$

with

$$0 \leq u_i \leq 1, \quad i = 2, \dots, M, \quad 0 \leq w u_1 \leq 1, \quad C_{ji} = u_{i+1} \dots u_j.$$

Because of the divergence of $[f(-w)]^{-3}$ near $w = 1$, we need as previously to construct a counter term $\tilde{F}_{1,N}^{(M)}$ to make the amplitude finite. To do that, as previously, we replace in expression (5.1) the function

$$\Psi_N(x, w) = -\frac{2 \ln w}{\pi} \sin \left(\frac{1}{2} \pi \frac{\ln x}{\ln w} \right) \prod_{n=1}^{\infty} \frac{(1 - 2q_N^{2n} \cos \pi(\ln x)/(\ln w) + q_N^{4n})}{(1 - q_N^{2n})^2}, \quad (5.2)$$

(where $q_N = -i \exp \pi^2/2 \ln w$) by

$$\tilde{\Psi}_N(x, w) = -\frac{2 \ln w}{\pi} \sin \frac{1}{2} \pi \frac{\ln x}{\ln w}$$

and replace the integration measure by the function

$$(1+w)f(-w)^{-3} w^{-2} (\ln w)^{-2 + \gamma'(w)},$$

where $\gamma'(w)$ is arbitrary.

In this way we define $\tilde{F}_{1,N}^{(M)}$. Let us now make the following change of variables:

$$\theta_j = \pi \frac{\ln u_1 u_2 \dots u_j}{\ln w} + \pi, \quad j = 1, 2, \dots, N-1,$$

$$w = u_1 \dots u_N$$

The integration region is now

$$0 \leq \theta_1 \leq \dots \leq \theta_{N-1} \leq 2\pi$$

and we get

$$\tilde{F}_{1,N}^{(M)} = g^2 C_{2,N} \mathcal{F}_1^{(M)}(p_1, \dots, p_M),$$

where

$$C_{2,N} = -4\pi^3 \int_0^1 dw h'(w),$$

$$h'(w) = (1+w)f(-w)^{-3} w^{-2} (\ln w)^{-3} + \gamma'(w)(\ln w)^{-1}.$$

We see that the counter term associated with the non-orientable loop is proportional to the planar counter term. In the case of one-loop diagrams, one knows that a set of unitary diagrams is obtained by taking all graphs which are not equivalent by duality transformation, each with equal weight. So, at least at order g , we can simply add the vectors tadpole T and T_N which reproduce, when inserted on a tree, the planar and the non-orientable loop. In the Virasoro case, the sum of these two vectors can be renormalized using a single constant, since the two counter terms are proportional to the vector $\tilde{t}$ defined previously. The renormalized sum is:

$$(T + T_N)_R = T + T_N - (C_2 + C_{2,N})\tilde{t}$$

and involves only one arbitrary constant, $C_2 + C_{2,N}$.

Therefore we can conclude that at least for the one-loop planar and non-orientable graphs, the renormalization in the Virasoro case depends on a single constant.

We do not deal here with the non-planar orientable diagrams, as they can as a rule be defined by analytic continuation in the external momenta. However, in this way one gets an unphysical cut violating unitarity. Unless some averaging procedure can be found to reconcile the expression of this graph with unitarity, no solution is seen to this problem.

6. ITERATION OF THE PLANAR TADPOLE IN THE VIRASORO CASE

This section will be rather formal, as we do not know yet a practical way of defining and handling the operator $P(p)$ off-mass-shell. However, in principle it is feasible and one must hope that the problem will be solved. The problem of defining a fully projected propagator is not confined to the renormalization procedure. On the contrary one should say that as soon as such an object is defined, the corresponding counter terms will also be defined

We can state formally that the operators $\Sigma(p)$ and T are perfectly suited to compute multiloop graphs, provided that

- (1) we replace in the integrand $f^{-4}(w)$ by $f^{-3}(w)$, and
 - (2) projected propagators should be used between two such operators.
- Similar rules apply to operator $\Sigma_T(p)$ (non-planar self-energy loop), and to the non-orientable tadpole operator.

When two-loop diagrams are computed, using renormalized tadpoles, the result is not finite, because the summation over intermediate states creates a new divergence [6].

This new divergence is in fact related to the duality transformations of fig. 4 and is nothing else than the divergence of the non-planar self-energy operator. Therefore, we can expect this 'third' divergence to be also present in the Virasoro case.

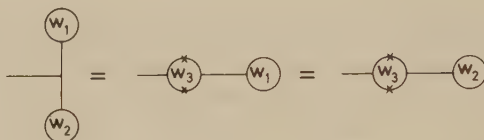


Fig. 4. Duality relations in the two-loop tadpole.

Thus the iteration of the planar renormalized tadpole ($T - C_2 \tilde{t}$), which has finite matrix elements at order g gives a new divergence at order g^3 when the two-loop tadpole is computed.

One has thus to introduce new counter terms specifically built for the two-loop tadpole, to cancel this new divergence. Such counter terms have been built by Cremmer and Neveu [7]. What we wish to show is that in the Virasoro case, to order g^4 , the introduction of these new counter terms amounts to replace the constant C_2 by $C_2 + g^2 C_4$, and therefore we do not introduce a new arbitrariness at order g^3 , but still have only one arbitrary constant. We just give the outline of the proof and refer to [7] for notation.

The two-loop tadpole is designated by $T_2(w_1, w_2, w_3)$, where w_1, w_2 refer to the integration variables of each of the tadpoles. The quantity w_3 to the third divergence exhibited by the duality transformation of fig. 4.

The iteration of the counter term tadpole gives rise to three terms referred to as $-\tilde{T}_2(w_1, w_2, w_3)$, $-\tilde{T}_2(w_1, w_2, w_3)$, $\tilde{T}_2(w_1, w_2, w_3)$. In the Virasoro case, these are respectively proportional to $\overline{C_2}, \overline{C_2}, C_2^2$.

However the sum of these four terms is given by a divergent integral near $w_3 = 1$. Therefore one introduces four additional terms, such as the divergence at $w_3 = 1$ is cured while keeping finiteness at $w_1 = 1$ and $w_2 = 1$.

Let us see for instance how $-T_2(w_1, w_2, w_3)$ is defined. $T_2(w_1, w_2, w_3)$ can be expressed by the duality transformation of fig. 4 as

$$T_2(w_1, w_2, w_3) = \Sigma_T(w_3) P^+ \mathcal{D} P T^+(w_1) = \Sigma_T(w_3) P^+ \mathcal{D} P T^+(w_2). \quad (6.1)$$

Therefore to cancel the divergence near $w_3 = 1$ we introduce the counter term

$$\tilde{T}_2(w_1, w_2, w_3) = \tilde{\Sigma}_T(w_3) P^+ \mathcal{D} P T^+(w_1) = \tilde{\Sigma}_T(w_3) P^+ \mathcal{D} P T^+(w_2). \quad (6.2)$$

By defining an operator $\tilde{\Sigma}_T$ such as $\Sigma_T - \tilde{\Sigma}_T(w_3)$ is finite near $w_3 = 1$. (We recall that it is not possible to define such an operator for any momentum p having this property, but here we are dealing with $p = 0$.)

At this stage, one uses a fundamental property of the operator Σ_T . This operator has the following form at $p = 0$:

$$\Sigma_T(0) = 4\pi^2 g^2 \int_0^1 d\mu(w_3) \int_{w_3}^1 \frac{dx}{x} \langle O_{1,2} | \exp \Delta, \quad (6.3)$$

$$\Delta = (a_1 | (1 - w_3) E_T(1 - w_3) | a_2) + (a_1 | (1 - w_3) E(1 - w_3) | a_1) \\ + (a_2 | (1 - w_3) E(1 - w_3) | a_2),$$

$$d\mu(w_3) = w_3^{-2} [f(w_3)]^{-3} (\ln w_3)^{-2} (1 - w_3)^{-1}. \quad (6.4)$$

The matrix $E_T(x, x_3)$ has the fundamental property that

$$E_T(x, x_3)_{q_3 \rightarrow 0} = O(q_3^2), \quad q_3 = \exp \left(\frac{2\pi^2}{\ln w_3} \right). \quad (6.5)$$

Therefore in this limit, the oscillators a_1, a_2 are fully decoupled. This property is transparent on the quark diagram corresponding to Σ_T , where the quark line can be pulled apart (fig. 5).



Fig. 5. Quark diagram of the non-planar self-energy operator.

Choosing line 1 to go to the external side of the diagram, line 2 going to the tadpole, ending in vacuum, we choose to define Σ_T by replacing Δ by $\tilde{\Delta}$ where

$$\tilde{\Delta} = (a_1 | (1 - w_3) \tilde{E} (1 - w_3) | a_1) + (a_2 | (1 - w_3) E (1 - w_3) | a_2) . \quad (6.6)$$

Defining now

$$\langle O_2 | \exp[(a_2 | (1 - w_3) E (1 - w_3) | a_2)] P^+ \mathcal{D} T^+(a_2) = g \lambda(w_3) , \quad (6.7)$$

(the oscillators a_2, a_1 being decoupled, this matrix element depends only on w_3) $\tilde{T}_1(w_1, w_2, \underline{w_3})$ has the following form:

$$\begin{aligned} & \tilde{T}_2(w_1, w_2, \underline{w_3}) \\ &= -4\pi^2 g^3 \int_0^1 d\mu(w_3) \ln w_3 \lambda(w_3) \langle O_1 | \exp(a_1 | (1 - w_3) \tilde{E} (1 - w_3) | a_1) . \end{aligned} \quad (6.8)$$

We thus see that $\tilde{T}_2(w_1, w_2, \underline{w_3})$ is nothing other than a counter term tadpole except it is of order g^3 instead of order g . Until now, this is independent, whether $\alpha_0 = 1$ or not.

Specifically in the case $\alpha_0 = 1$, we can now rewrite, using projection operators:

$$\tilde{T}_2(w_1, w_2, \underline{w_3}) P(0) = g^2 C_4^{(1)} \tilde{t} P(0) , \quad (6.9)$$

with

$$C_4^{(1)} = -8\pi^3 \int_0^1 d\mu(w_3) \lambda(w_3) (1 - w_3) , \quad (6.10)$$

where $\tilde{t}$ is the vector defined in sect. 5.

One can similarly see that the other three additional counter terms give contributions of exactly the same type, with constants called $C_4^{(2)}$, $C_4^{(3)}$ and $C_4^{(4)}$. If

$$C_4 = \sum_{i=1}^4 C_4^{(i)} ,$$

then the sum of the four additional counter terms is nothing other than $-g^2 C_4 \tilde{t} P(0)$.

Therefore, if instead of having chosen as the original counter term tadpole vector $C_2 \tilde{t}$ we had chosen the vector $(C_2 + g^2 C_4) \tilde{t}$ the iteration of the tadpole $T - (C_2 + g^2 C_4) \tilde{t}$ would give finite results for the sum of the one-loop and the two-loop tadpole, provided that we only keep the terms of order g^3 in consideration.

Of course, C_4 is not determined entirely when C_2 is chosen. This is because we can as usual change the integration measure $d\mu(w_3)$ to define

$\tilde{\Sigma}_T$. However, as only the sum of the constants C_2 and $g^2 C_4$ occurs, we can say as before there is only one arbitrary constant, which is the finite part of the infinite constant $C_2 + g^2 C_4$.

So, in the Virasoro case, the counter terms introduced to cancel the additional divergences of the two-loop diagram can be reinterpreted as a change of the infinite constant in the factor of the original counter term.

7. CONCLUSION

We have re-examined the arbitrariness of the renormalization of the planar and non-orientable loops. It turns out that in the Virasoro case of unit intercept the arbitrariness can be recast into only one multiplicative constant. It is however not obvious how the result can be interpreted in terms of quantum-field-theory renormalization, since one does not know yet how renormalized propagators and vertices should be defined, because of the counting problem.

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MASS-SHIFTS IN SECOND ORDER OF PERTURBATION THEORY
IN THE DUAL RESONANCE MODEL*

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Abstract : It is shown that in the Virasoro case of unit intercept, the divergent part at order g^2 of the mass matrix of the physical states is simply proportional to the unit matrix. Therefore it is possible in this case to compute all mass-shifts at order g^2 up to a single subtraction constant.

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It has been shown that the renormalization procedure devised for the one-loop planar graphs of the dual resonance model, while keeping the properties of perturbative unitarity and duality, is highly non-unique in the case of arbitrary intercept as an arbitrary function $g(w)$ can be introduced in the integrand defining the counterterm. That such a non-uniqueness is intimately connected with the problem of ghosts and spurious states is shown by the fact that in the Virasoro case [1], where ghosts can be (presumably) eliminated by the infinite number of gauge-like identities, the integration on the variable w factors out from the counterterm after a suitable change of variables therefore leaving only an arbitrary multiplicative constant in the counterterm : (see ref. [2])

$$\tilde{F}_1^{(N)}(p_1 \dots p_N) = g^2 C \mathcal{F}_1^{(N)}(p_1 \dots p_N) \quad \text{eq. (1)}$$

$$\text{where } C = -8\pi^3 \int_0^1 dw [w^{-2} f^{-3}(w) (1-w) (\ln w)^{-3}] + C^F \quad \text{eq. (2)}$$

C^F is an arbitrary finite constant, $f(w)$ is the partition function ; for the expression of $\mathcal{F}_1^{(N)}$ see equation (4) below.

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MASS-SHIFTS IN SECOND ORDER OF PERTURBATION THEORY
IN THE DUAL RESONANCE MODEL*

J. SCHERK

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Abstract : It is shown that in the Virasoro case of unit intercept, the divergent part at order g^2 of the mass matrix of the physical states is simply proportional to the unit matrix. Therefore it is possible in this case to compute all mass-shifts at order g^2 up to a single subtraction constant.

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It has been shown that the renormalization procedure devised for the one-loop planar graphs of the dual resonance model, while keeping the properties of perturbative unitarity and duality, is highly non-unique in the case of arbitrary intercept as an arbitrary function $g(w)$ can be introduced in the integrand defining the counterterm. That such a non-uniqueness is intimately connected with the problem of ghosts and spurious states is shown by the fact that in the Virasoro case [1], where ghosts can be (presumably) eliminated by the infinite number of gauge-like identities, the integration on the variable w factors out from the counterterm after a suitable change of variables therefore leaving only an arbitrary multiplicative constant in the counterterm : (see ref. [2])

$$\tilde{F}_1^{(N)}(p_1 \dots p_N) = g^2 C \mathcal{F}_1^{(N)}(p_1 \dots p_N) \quad \text{eq. (1)}$$

$$\text{where } C = -8\pi^3 \int_0^1 dw [w^{-2} f^{-3}(w) (1-w) (\ln w)^{-3}] + C^F \quad \text{eq. (2)}$$

C^F is an arbitrary finite constant, $f(w)$ is the partition function ; for the expression of $\mathcal{F}_1^{(N)}$ see equation (4) below.

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Divergence of the Two-Loop Planar Graph in the Dual-Resonance Model*

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The two-loop planar tadpole is constructed by inserting two projected planar tadpoles on a symmetric three-Reggeon vertex. The determinant, which is the generalization of the partition function appearing in the one-loop diagram, is studied as a function of the integration variables of the loops and propagators. It is found that although the determinant contains an infinite number of possibly exponentially divergent factors, only three of them actually diverge. The first two divergent factors are those associated with each tadpole. The additional divergence is associated with the closed path going around the two loops which does not cross itself. The calculation is also performed with the Koba-Nielsen variables of the original N -loop graph, and the same result is derived.

I. INTRODUCTION

It has been observed recently that any Feynman-like diagram of the dual-resonance model can be built from trees and from four primitive operators.¹ All planar graphs, for instance, can be constructed by iteration of the planar one-loop tadpole. The problem of eliminating the contribution of spurious states has been solved, and the one-loop planar projected tadpole, attached to a projected propagator, has been obtained.² By using the methods developed by Kaku and Yu,^{3,4} exponentials of quadratic forms in the creation and annihilation operators can be handled to compute the two-loop planar tadpole, obtained by putting two tadpoles on a symmetric three-Reggeon vertex. The determinant which appears in the calculation has a structure similar to the partition function encountered in the one-loop diagram computation.

The determinant is of the form

$$\prod_{n=1}^{\infty} \prod_{(P)} (1 - W_P^n)^{-4},$$

where W_P is associated with the projective transformation P . This number is called the multiplier of the transformation. The set (P) is a subset of the group of transformations generated by two transformations P_1 and P_2 . The set contains elements such as P_1 , P_2 , and $P_1^{\alpha_1} P_2^{\beta_1} P_1^{\alpha_2} \dots P_1^{\alpha_n} P_2^{\beta_n}$. In this set, the exponents $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are negative or positive nonzero integers chosen with the restriction that a given transformation P^n occurs only once in the product over P . For instance, $P_1 P_2 P_1 P_2$ is included in the term associated with $P_1 P_2$. [More restrictions on (P) and W_P are given below.]

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² D. J. Gross, and J. H. Schwarz, Phys. Rev. Letters **25**, 406 (1970).

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⁴ M. Kaku and Loh-Ping Yu, Berkeley report (unpublished).

In principle, each factor in $\prod_n \prod_{(P)} (1 - W_P^n)^{-4}$ can become an exponentially divergent factor. However, we show that, because of linear dependences,

$$0 \leq W_P \leq 1.$$

Also, the equality $W_P = 1$ can be satisfied only if $P = P_1, P_2$, or $P_1 P_2$. All other transformations contribute a finite factor. The divergent factor is thus

$$\left[\prod_{n=1}^{\infty} (1 - W_1^n)(1 - W_2^n)(1 - W_{P_1 P_2}^n) \right]^{-4}.$$

The first two factors are associated with each of the tadpoles. In terms of KSV⁵ lines, one sees that the three divergent factors correspond to closed paths which do not cross themselves, while the finite factors correspond to self-crossing closed paths.

If linear dependences are neglected, then it is possible to have $|W_P| = 1$. We see, therefore, that in the limit $W_1 = W_2 = 1$, any of the factors in $\prod_n \prod_{(P)} (1 - W_P^n)^{-4}$ may diverge. Thus, linear dependences play a much more important role in the two-loop than in the one-loop case.

There is a simple interpretation of this result. The generators P_1 and P_2 depend on four variables, including W_1 and W_2 . In the limit where W_1 (or W_2) goes to one, the transformation P_1 (or P_2) does not go to the identity but degenerates into a parabolic transformation whose invariant points are identical. This is ensured by linear dependences. If one neglected linear dependences, then P_1 and P_2 would go to one in the limit $W_1 = W_2 = 1$. Then any transformation in the group would also become the identity transformation, and so each factor in $1 - W_P$ would go to zero. In the single-loop calculation, linear dependences merely added a factor $1 - W$. In higher-loop calculations, we have complicated constraints on the variables in the singularity.

In Sec. II, we give some identities for projective transformations and discuss the structure of the determinant. In Sec. III, we perform the computation of the two-loop tadpole and identify the projective trans-

⁵ K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969).

formations involved. We also study the divergence of the determinant and prove that $0 \leq W_{P_1^n P_2^m} \leq 1$. In Sec. IV, we extend this result by a geometrical method to any transformation in the group. In Appendix A, we give the proof in terms of the variables used earlier and show that the inequality which ensures $0 \leq W_P \leq 1$ is satisfied. A similar proof is in Appendix B in terms of the Koba-Nielsen variables appearing in the N -loop computation of Ref. 4.

II. IDENTITIES FOR PROJECTIVE TRANSFORMATIONS

A very good account of the properties of projective transformations can be found in Ford.⁶ We will now give the properties and definitions used in this paper.

A projective transformation is a transformation defined by

$$P(z) = \frac{az+b}{cz+d} = z'.$$

We assume that $ad-bc=1$. Projective transformations can be classified by the value of their multiplier, as we shall see. It is convenient to reexpress P in the form

$$P: \frac{z'-\zeta}{z'-\zeta'} = W_P^{-1} \frac{z-\zeta}{z-\zeta'}. \quad (2.1)$$

ζ and ζ' are the invariant points of the transformation [i.e., $P(\zeta)=\zeta$ and $P(\zeta')=\zeta'$]. W_P is called the multiplier. P^n has the following form:

$$\frac{z'-\zeta}{z'-\zeta'} = (W_P^{-n}) \frac{z-\zeta}{z-\zeta'}. \quad (2.2)$$

If W_P is real, one can always choose $0 \leq W_P \leq 1$. The transformation is then called hyperbolic. If $|W_P|=1$, it is called elliptic; if W_P is complex, it is called loxodromic.

If $\zeta=\zeta'$, the transformation is called parabolic. It is then written as

$$\frac{1}{z'-\zeta} = \frac{1}{z-\zeta} + \delta.$$

For reasons which will become clearer later, we consider a parabolic transformation as a degenerate case of a hyperbolic transformation whose multiplier goes to 1.

The multiplier W satisfies the following equation:

$$(W_P)^{1/2} + (W_P)^{-1/2} = \text{Tr}(P),$$

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad-bc=1. \quad (2.3)$$

W has the following properties:

- (a) $W_{P^n} = (W_P)^n$,
- (b) $W_{P_1 P_2} = W_{P_2 P_1}$,

- (c) $W_{P^{-1}} = W_P$ (Ref. 7),
- (d) $W_{GPG^{-1}} = W_P$.

The divergent factor $\prod_n (1-W^n)^{-4}$ found in the one-loop diagram computation has a simple projective interpretation: If $P(z)=Wz$, then $W_P=W$. Since the W appearing in the single-loop singularity is just the multiplier of P , we can now write $\prod_{[P]} (1-W_P)$, where $[P] = [P^i, i=1, \infty]$.

We thus expect the divergent factor of the two-loop diagram to be of the form $\prod_n \prod_{(P)} (1-W_P^n)^{-4}$, P being an element of the group generated by two transformations P_1 and P_2 such that cyclic permutations and inverses of such permutations of previous elements are not double counted. We also exclude those elements which can be written as a power of previous elements. This result easily generalizes to the N -loop case.⁸

We notice that the two transformations P_1, P_2 are hyperbolic transformations, and can be shown to be real. The group generated by P_1, P_2 will thus leave the real axis invariant; this group is called Fuchsian (see Ford⁶). Since the transformations in the group are real, either $|W_P|=1$ or W_P is real, in which case one can always choose $0 \leq W_P \leq 1$.

The KSV interpretation of a factor like

$$1 - (W_{P_1^{\alpha_1} P_2^{\beta_1} \dots P_1^{\alpha_n} P_2^{\beta_n}})^k$$

is to say that it describes a closed path going α_1 times around the point corresponding to loop 1, β_1 around 2, and so on, the whole thing being iterated k times. We thus expect that this factor should at least be real and between zero and one, as in the case with one-loop diagrams. We suspect that all the transformations of the group generated by P_1 and P_2 must be hyperbolic.

Thus we are led to the following question: Given two real hyperbolic transformations, when can we say that any transformation in the group which they generate is also hyperbolic?

To solve this problem, we consider two real hyperbolic transformations:

$$P_1: \frac{z'-\zeta_1}{z'-\zeta'_1} = W_1^{-1} \frac{z-\zeta_1}{z-\zeta'_1},$$

$$P_2: \frac{z'-\zeta_2}{z'-\zeta'_2} = W_2^{-1} \frac{z-\zeta_2}{z-\zeta'_2}.$$

Forming their product, one can show that $W_{P_1 P_2}$ satisfies the following equation:

$$(W_{P_1 P_2})^{1/2} + (W_{P_1 P_2})^{-1/2} = \left| W_1 + W_2 + \frac{(\zeta'_1 - \zeta_2)(\zeta'_2 - \zeta_1)}{(\zeta'_1 - \zeta_1)(\zeta'_2 - \zeta_2)} (1 - W_1)(1 - W_2) \right| \times (W_1 W_2)^{-1/2}. \quad (2.4)$$

⁷ Equation (2.3) admits obviously both W_P and W_P^{-1} as roots. Therefore, in the case of real hyperbolic transformations it is convenient to choose the solution $0 \leq W_P \leq 1$. Thus property (a) is meant only for positive values of the integer n .

⁸ This conjecture is due to S. Mandelstam. It has been checked up to fourth order in Ref. 4 for the N -loop amplitude.

⁶ L. R. Ford, *Automorphic Functions* (Chelsea, New York, 1951).

We therefore see that $W_{P_1P_2}$ depends not only on the multipliers of P_1 and P_2 , but also on the cross ratio of the four invariant points of the two transformations. This equation can be checked easily in the limit where $W_1 \rightarrow 1$ or when the invariant points of the two transformations are the same (in this case $W_{P_1P_2} = W_1W_2$).

It is easy to see that $W_{P_1P_2} = 1$ does not necessarily mean that $P_1P_2 = I$. P_1P_2 can be of the parabolic type; in that case $W_{P_1P_2}$, as given by Eq. (2.4), is found to be 1. $W_{P_1P_2}$ satisfies the same equation if W_1, W_2 are replaced by W_1^n, W_2^m .

Some insight into the group properties of projective transformations will be obtained by introducing the isometric circle I_P of the transformation P . This circle is defined by $|cz+d|=1$; it satisfies the property $|dz'| = |dz|$. In terms of the multiplier W_P and the two invariant points, its center is given by

$$z = -\frac{d}{c} = -\frac{\zeta'W_P - \zeta}{1 - W_P} \quad (2.5)$$

and its radius by

$$R = \frac{1}{c} = \frac{(W_P)^{1/2}}{1 - W_P} (\zeta - \zeta'). \quad (2.6)$$

To the inverse transformation P^{-1} , we associate an isometric circle $I_{P^{-1}}$ with the same radius and with center

$$z' = \frac{a}{c} = \frac{\zeta' - \zeta W_P}{1 - W_P}. \quad (2.7)$$

The transformation P can be realized as an inversion with respect to I_P followed by a reflection on L , the perpendicular bisector of the line segment joining the centers of I_P and $I_{P^{-1}}$.

The most relevant geometrical property of the isometric circles associated with P and P^{-1} is the following:

Theorem 1. If P is a real transformation, then

I_P and $I_{P^{-1}}$ exterior to one another $\leftrightarrow P$ is hyperbolic,
 I_P and $I_{P^{-1}}$ tangent to one another $\leftrightarrow P$ is parabolic,
 I_P and $I_{P^{-1}}$ intersect each other $\leftrightarrow P$ is elliptic.

As we will study the linear transformations belonging to a group, it will be useful to use the two following theorems which relate the isometric circle of a product of two transformations to the isometric circles of each of the transformations.

Theorem 2. Let r_1, r_2 , and r_{12} be the radii of the isometric circles of P_1, P_2 , and P_1P_2 and z_1, z_2 , and z_{12} their centers. z_2' is the center of $I_{P_2^{-1}}$. Then

$$r_{12} = \frac{r_1 r_2}{|z_2' - z_1|}, \quad |z_{12} - z_2| = \frac{r_2^2}{|z_2' - z_1|}. \quad (2.8)$$

Theorem 3. Let $I_{P_1}, I_{P_2}, I_{P_2^{-1}}$, and I_u be the isometric circles of the linear transformations P_1, P_2 , and P_2^{-1} , and u ($u = P_1P_2$).⁹ If I_{P_1} and $I_{P_2^{-1}}$ are exterior to one another, then I_u is contained in I_{P_2} . If I_{P_1} and $I_{P_2^{-1}}$ are tangent, then $I_{P_1P_2}$ lies in I_{P_2} and is tangent internally.

We will use the last two theorems to find the location of the isometric circles of the group generated by two transformation P_1 and P_2 . We use Theorem 1 to deduce the type of each of the transformations in the group. We proceed now to the actual computation of the two-loop tadpole and to the identification of the projective transformations involved.

III. DETERMINANT OF TWO-LOOP TADPOLE

We use the following form of the tadpole operator with projected twisted propagator:

$$T\mathfrak{D} = 4\pi^2 g \int_0^1 \frac{dW dx}{\ln^2 W} (Wx)^{-\alpha_0-1} (1-x)^{\alpha_0-1} \times f(W)^{-4} \langle 0 | e^{(a|\Delta(W,x)|a)}, \quad (3.1)$$

where

$$\Delta(W, x) = (x) M_- \left(\frac{x-1}{x} \right) M_-^T(x) (1-W) \times E(1-W)(x) M_- \left(\frac{x-1}{x} \right) M_-^T(x).$$

For the notation, we refer to Ref. 2. Using the properties of E , one can also write

$$\Delta(W, x) = (x) M_- \frac{(1-x)(1-W)}{1-x(1-W)} E \frac{(1-x)(1-W)}{1-x(1-W)} \times M_-^T(x). \quad (3.2)$$

We now build the following two-loop tadpole $T_2(c)$ (see Fig. 1):

$$T_2(c) = \langle 0_{ab} | T(a) \mathfrak{D}(a) T(b) \mathfrak{D}(b) \exp[(a^\dagger | M_- | b^\dagger) + (b^\dagger | M_- | c^\dagger) + (c^\dagger | M_- | a^\dagger)] | 0_{ab} \rangle \\ = 16\pi^4 g^2 \int_0^1 \frac{dW_1 dW_2 dx_1 dx_2}{\ln^2 W_1 \ln^2 W_2} (W_1 W_2 x_1 x_2)^{-\alpha_0-1} \times [(1-x_1)(1-x_2)]^{\alpha_0-1} \mathcal{T}_2, \quad (3.3)$$

where

$$\mathcal{T}_2 = [f(W_1) f(W_2)]^{-4} \langle 0_{ab} | \exp[(a | \Delta_1 | a) + (b | \Delta_2 | b)] \times \exp[(a^\dagger | M_- | b^\dagger) + (b^\dagger | M_- | c^\dagger) + (c^\dagger | M_- | a^\dagger)] | 0_{ab} \rangle. \quad (3.4)$$

Introducing two complete sets of coherent states, one

⁹ Ford states incorrectly that $u = P_2P_1$.

can write

$$\mathcal{T}_2 = [f(W_1)f(W_2)]^{-4} \times \int \frac{d^2\partial_a d^2\partial_b}{4\pi^2} \exp\left(\partial^T \frac{\Gamma}{2} \partial + \partial^T B\right), \quad (3.5)$$

where $\partial^T = (\partial_a, \partial_a^*, \partial_b, \partial_b^*)$ must be understood symbolically as an infinite-dimensional vector. (Notice that most of the indices are suppressed for simplicity.)

Γ and B are the following (infinite) matrices:

$$\Gamma = \begin{pmatrix} 2\Delta_1 & -1 & 0 & 0 \\ -1 & 0 & 0 & M_- \\ 0 & 0 & 2\Delta_2 & -1 \\ 0 & M_-^T & -1 & 0 \end{pmatrix}, \quad \Gamma^T = \Gamma$$

$$B = \begin{pmatrix} 0 \\ M_-^T C^+ \\ 0 \\ M_- C^+ \end{pmatrix}.$$

Using the method of the principal axis,⁴ one gets

$$\mathcal{T}_2 = [f(W_1)f(W_2)(\det\Gamma)^{1/2}]^{-4} \exp(-\frac{1}{2}\partial^T \Gamma^{-1} \partial). \quad (3.6)$$

The exponential goes to unity when the external state is the vacuum. We can identify the factor raised to the fourth power as the divergent part. We call

$$D_2 = [f(W_1)f(W_2)(\det\Gamma)^{1/2}]^{-1}$$

the determinant of the two-loop planar tadpole.

One can write $\det\Gamma = \det(1-A)$, where

$$A = \begin{pmatrix} 0 & 2\Delta_1 & 0 & 0 \\ 0 & 0 & M_- & 0 \\ 0 & 0 & 0 & 2\Delta_2 \\ M_-^T & 0 & 0 & 0 \end{pmatrix}.$$

Using the fact that

$$\det(1-A) = \exp\left(-\sum_{n=1}^{\infty} \frac{\text{Tr} A^n}{n}\right),$$

we see that A , A^2 , and A^3 do not contribute to the trace. But A^4 is diagonal, and hence only powers of A which are multiples of 4 contribute to the sum. Hence, we find

$$D_2^{-1} = f(W_1)f(W_2) \times \exp\left(-\sum_{n=1}^{\infty} 4^n \frac{\text{Tr}(\Delta_1 M_- \Delta_2 M_-^T)^n}{2n}\right). \quad (3.7)$$

The interpretation of the infinite product is that it corresponds to going from tadpole 2 to tadpole 1 (undotted-dotted), then from tadpole 1 back to tadpole 2 (dotted-undotted). One thus makes a cyclic path which can be iterated n times. This is consistent with the picture given by Kikkawa and Sato.¹⁰

¹⁰ K. Kikawa and H. Sato, University of Tokyo report (unpublished).

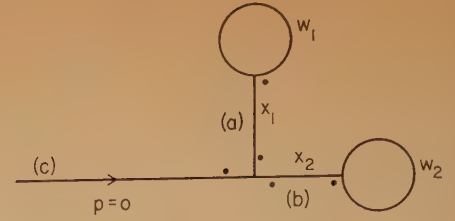


FIG. 1. Two-loop planar tadpole.

We proceed now to the computation of the trace of order 1:

$$(E)_{mn} = \frac{1}{2(mn)^{1/2} \ln W} + \frac{1}{2} \left[M_+ \frac{(W)}{1-(W)} M_- + M_-^T \frac{(W)}{1-(W)} M_+ \right]_{mn}. \quad (3.8)$$

We drop the first term, which comes from the integration over d^4k . This is legitimate because the integration over d^4k is not supposed to introduce new divergences. Dropping it amounts to not performing the integration.

We can then write

$$E = \frac{1}{2} \sum_{n=1}^{\infty} [M_+(W^n)M_- + M_-^T(W^n)M_+] = \frac{1}{2} \sum_{n=-\infty, n \neq 0}^{+\infty} M_+(W^n)M_- = \frac{1}{2} \sum_{n \neq 0} \frac{1}{1-W^n} M_+ \frac{-W^n}{1-W^n}. \quad (3.9)$$

Thus

$$4 \text{Tr}(\Delta_1 M_- \Delta_2 M_-^T) = \sum_{n, m \neq 0} \Delta_{mn},$$

where

$$\Delta_{mn} = \text{Tr} \left[(a) M_- \left(\frac{b}{1-W_1^n} \right) M_+ \left(\frac{-bW_1^n}{1-W_1^n} \right) M_-^T \times (a) M_- \left(\frac{c}{1-W_2^n} \right) M_+ \left(\frac{-cW_2^n}{1-W_2^n} \right) M_-^T \right]$$

and

$$a = x_1(1-x_2), \quad b = \frac{(1-x_1)(1-W_1)}{1-x_1(1-W_1)}, \quad c = \frac{-x_2(1-W_2)}{1-x_2(1-W_2)}.$$

This trace can be reduced to a trace of the form

$$\Delta_{mn} = \text{Tr}(\alpha M_+ \beta M_+ \gamma M_+ \delta M_+) = \frac{1}{2} \left[\frac{1}{(1-4u_{mn})^{1/2}} - 1 \right], \quad (3.10)$$

where

$$u_{mn} = \alpha\beta\gamma\delta / [1 - \alpha - \beta - \gamma - \delta + \alpha\gamma + \delta\beta]^2.$$

After some tedious but straightforward computations, one finds that u can be recast in the form

$$u_{mn} = \frac{W_1^n W_2^n}{[\rho(1-W_1^n)(1-W_2^n) + W_1^n + W_2^n]^2}. \quad (3.11)$$

ρ is a function of W_1, W_2, x_1 , and x_2 but is independent of n and m . It is given by

$$\rho = - \frac{[1 - x_1(1 - x_2W_1)][(1 - x_1)(1 - x_2) + x_2W_2]}{x_1x_2(1 - x_1)(1 - x_2)(1 - W_1)(1 - W_2)}. \tag{3.12}$$

Let us now define $W_{P_1^n P_2^m}$ by the following equation:

$$u_{mn} = [(W_{P_1^n P_2^m})^{1/2} + (W_{P_1^n P_2^m})^{-1/2}]^{-2}. \tag{3.13}$$

Then we see that $W_{P_1^n P_2^m}$ satisfies Eq. (2.4), where W_1 and W_2 are replaced by W_1^n and W_2^m . Remarkably, ρ is independent of n and m and is the cross ratio of the four invariant points of two transformations P_1 and P_2 such that $W_{P_1} = W_1$ and $W_{P_2} = W_2$.

We choose $0 \leq W_{P_1^n P_2^m} \leq 1$ (provided that it is possible, which we show below). We find

$$\Delta_{mn} = W_{P_1^n P_2^m} / (1 - W_{P_1^n P_2^m});$$

and one can argue that D_2^{-1} contains, besides $\prod_n (1 - W_{P_1^n})(1 - W_{P_2^n})$, the factor

$$[\prod_{n,m} (1 - W_{P_1^n P_2^m})]^{1/2}.$$

Here we have used the fact that

$$\prod_n (1 - x^n) = \exp \left[- \sum_n \frac{x^n}{n(1 - x^n)} \right]. \tag{3.14}$$

In conclusion, the trace of order 1 provides us with the factor

$$\begin{aligned} &\exp \left[- \sum_{m,n \neq 0} \frac{W_{P_1^n P_2^m}}{2(1 - W_{P_1^n P_2^m})} \right] \\ &= \exp \left[- \sum_{m,n \in c} \frac{W_{P_1^n P_2^m}}{1 - W_{P_1^n P_2^m}} \right], \end{aligned} \tag{3.15}$$

where the class c is defined such that only one of the two factors $P_1^n P_2^m$ and $P_1^{-n} P_2^{-m}$ appears in c .

The relation between W_1^n, W_2^m , and $W_{P_1^n P_2^m}$ is exactly the one which was anticipated in Sec. II. We thus deduce that the group of projective transformations associated with the two-loop tadpole is generated by two transformations P_1 and P_2 , such that

$$\begin{aligned} W_{P_1} &= W_1, \quad W_{P_2} = W_2, \\ \rho(W_1, W_2, x_1, x_2) &= (\zeta_1' - \zeta_2)(\zeta_2' - \zeta_1) / (\zeta_1' - \zeta_1)(\zeta_2' - \zeta_2). \end{aligned} \tag{3.16}$$

We cannot expect to gain further information from the computation of higher-order terms,¹¹ since two real projective transformations are functions of six parameters, but we can always perform a similarity transformation on P_1 and P_2 depending on three parameters which will leave W_P invariant. The determinant is thus a function of W_1, W_2 , and ρ .

We now examine the question of knowing if $0 \leq W_{P_1^n P_2^m} \leq 1$ and when it reaches 1. It would be, of course, very unpleasant to have $W_{P_1^n P_2^m} = 1$, because it would make $\prod_r (1 - W_{P_1^n P_2^m r})^{-4}$ meaningless. The question amounts to knowing if $0 \leq u_{mn} \leq \frac{1}{4}$ and when it reaches $\frac{1}{4}$.

We first notice, from Eq. (3.11), that if ρ were independent of W_1, W_2 , then $u = \frac{1}{4}$ would be true near $W_1 = W_2 = 1$. However, this is not realized, due to the W_1, W_2 dependence of ρ . This reflects the fact that the matrix E , which is singular when $W \rightarrow 1$, is sandwiched between two $(1 - W)$'s, which makes it regular in this neighborhood. Thus it is not obvious at all that $W_{P_1^n P_2^m}$ reaches 1.

In general, we can ask the following question: Given two transformations P_1 and P_2 with multipliers W_1 and W_2 (such that $0 \leq W_1, W_2 \leq 1$), and the cross ratio ρ of their four invariant points, what is the necessary and sufficient condition on ρ which ensures that $0 \leq W_{P_1 P_2} \leq 1$?

We see that

$$\begin{aligned} 0 \leq W_{P_1 P_2} \leq 1 &\leftrightarrow 0 \leq u_{11} \leq \frac{1}{4} \\ &\leftrightarrow 4W_1W_2 \leq |\rho(1 - W_1)(1 - W_2) + W_1 + W_2|^2 \\ &\leftrightarrow \left[-\rho - \frac{W_1 + W_2 + 2(W_1W_2)^{1/2}}{(1 - W_1)(1 - W_2)} \right] \left[-\rho - \frac{W_1 + W_2 - 2(W_1W_2)^{1/2}}{(1 - W_1)(1 - W_2)} \right] \geq 0 \\ &\leftrightarrow \text{(a) } -\rho \geq \frac{W_1 + W_2 + 2(W_1W_2)^{1/2}}{(1 - W_1)(1 - W_2)} \quad \text{or} \quad \text{(b) } -\rho \leq \frac{W_1 + W_2 - 2(W_1W_2)^{1/2}}{(1 - W_1)(1 - W_2)}. \end{aligned} \tag{3.17}$$

¹¹ Computation of the momentum-dependent terms provides a complete determination of the projective operators P_1 and P_2 . We apply the two-loop tadpole to a multiperipheral tree. Then one can show that the term raised to the power $k_i \cdot k_j$ contains the factor $\prod_{n=-\infty}^{+\infty} (\rho_i - P_1^n \rho_j)(\rho_i - P_2^n \rho_j)$; this gives a complete determination of P_1 and P_2 . One finds that the invariant points are given by

$$\begin{aligned} \zeta_1 &= \frac{1}{1 - x_1}, \quad P_1 = P_1(x_1, W_1): \zeta_1' = \frac{1 - x_1(1 - W_1)}{1 - x_1}, \\ \zeta_2 &= -\frac{1 - x_2}{x_2}, \quad P_2 = P_2(x_2, W_2): \zeta_2' = -\frac{1 - x_2}{x_2 W_2} \end{aligned}$$

and one checks that ρ is the cross ratio of these four invariant points. We see that due to linear dependences, these invariants points depend on W_1, W_2 . Also, when W_1 (or W_2) goes to one, $\zeta_1 = \zeta_1'$ (or $\zeta_2 = \zeta_2'$). Thus P_1 and P_2 degenerate into parabolic transformations. If this were not the case, then in the limit $W_1 = W_2 = 1$, one would have $P_1 = P_2 = I$; thus any element of the group generated by P_1, P_2 would go to I , and all the W_P would go to 1. Then the divergence would be much worse than in the one-loop case. The fact that P_1 and P_2 become parabolic in the limit prevents such an unpleasant situation.

So the necessary and sufficient condition for $0 \leq W_{P_1 P_2} \leq 1$ is that ρ satisfies either the inequality (a) or (b). We show in Appendix A that by minimizing the function $-\rho(x_1 x_2 W_1 W_2)$ with respect to x_1 and x_2 , the inequality (a) is satisfied.

Thus $-\rho$ satisfies

$$-\rho \geq [W_1 + W_2 + 2(W_1 W_2)^{1/2}] / (1 - W_1)(1 - W_2), \quad (3.18)$$

which is the necessary and sufficient condition for $0 \leq W_{P_1 P_2} \leq 1$. The fact that ρ is negative means that the invariant points of P_1 and P_2 are exterior to each other.

Once we know that $-\rho$ satisfies inequality (a), it is easy to show that $0 \leq W_{P_1^n P_2^m} \leq 1$, where n and m are not both equal to 1. Let us first show it for $n > 0$ and $m < 0$. In this case, $\rho(1 - W_1^n)(1 - W_2^m) \geq 0$. We thus have

$$u_{nm} \leq W_1^n W_2^m / [W_1^n + W_2^m]^2 \leq \frac{1}{4} \leftrightarrow 0 \leq W_{P_1^n P_2^m} \leq 1.$$

The case $n > 0, m > 0$ is slightly more difficult; we write

$$u_{mn} = W_1^{n-1} W_2^{m-1} \times \left[\frac{\rho(1 - W_1)(1 - W_2) + W_1 + W_2}{\rho(1 - W_1^n)(1 - W_2^m) + W_1^m + W_2^m} \right]^2 u_{11}. \quad (3.19)$$

Studying the square brackets as a linear function of ρ , one finds

$$u_{mn} \leq W_1^{n-1} W_2^{m-1} \left(\frac{1 - W_1}{1 - W_1^n} \frac{1 - W_2}{1 - W_2^m} \right) u_{11} \leq u_{11} \leq \frac{1}{4}. \quad (3.20)$$

Thus $0 \leq W_{P_1^n P_2^m} \leq 1$. We recall that $W_{P_1^n P_2^m} = W_{P_1^n P_2^m}$, so now all the cases have been exhausted. We have thus shown that all the transformations in the group of the form $P = P_1^n P_2^m$ are of the hyperbolic type or of the parabolic type. We show now a stronger result by seeing whether the equality $W_P = 1$ can actually be satisfied.

In the case $n > 0$ and $m > 0$ both different from one, we see that

$$u_{mn} \leq \frac{1}{4} W_1^{n-1} W_2^{m-1} \left(\frac{1 - W_1}{1 - W_1^n} \frac{1 - W_2}{1 - W_2^m} \right) < \frac{1}{4}, \quad (3.21)$$

as each of the factors is smaller than 1. For $W_1 = W_2 = 1$ we see that

$$u_{nm} \leq \frac{1}{4} (1/n^2 m^2) < \frac{1}{4}. \quad (3.22)$$

In the case $n > 0$ and $m < 0$, $u_{nm} = \frac{1}{4}$ can be realized only for $W_1 = W_2 = 1$. We see that for this value,

$$u_{nm} \leq 1/(-4nm + 2)^2 < \frac{1}{4}.$$

The case $n = m = 1$ is more involved. It is shown in Appendix A that in the limit

$$x_2 \rightarrow 0, \quad 1 - x_1 \rightarrow 0, \quad (1 - x_1)/x_2 \rightarrow (W_1 W_2)^{1/2}, \quad (3.23)$$

we have $u_{11} \rightarrow \frac{1}{4}$, and therefore $W_{P_1 P_2} \rightarrow 1$.

We have thus proved the following: Owing to the inequality

$$-\rho \geq [W_1 + W_2 + 2(W_1 W_2)^{1/2}] / (1 - W_1)(1 - W_2),$$

the transformations $Q = P_1^n P_2^m$ are such that $W_Q \leq 1$. For n, m not both equal to 1, the inequality is a strict one, so the factor $\prod_n (1 - W_Q^n)^{-4}$ is always finite in the integration region. For $n = m = 1$, $W_Q = 1$ can be satisfied without having W_1 or W_2 reach 1, so we get an extra divergent factor.

In Sec. IV, we extend this result to all the transformations of the group generated by P_1 and P_2 .

IV. GEOMETRICAL INTERPRETATION

We are looking for a geometrical interpretation of inequality (3.18) in order to generalize the statement $0 \leq W_P \leq 1$ to any transformation of the group generated by P_1 and P_2 .

To each of the transformations $P_1, P_1^{-1}, P_2, P_2^{-1}$, we associate the isometric circles $I_{P_1}, I_{P_1^{-1}}, I_{P_2}$, and $I_{P_2^{-1}}$. Since P_1 is hyperbolic, the circles I_{P_1} and $I_{P_1^{-1}}$ are external to each other (the same applies for I_{P_2} and $I_{P_2^{-1}}$). The centers of all these circles lie on the real axis.

We can perform on P_1, P_2 a similarity transformation ($P' = Q^{-1} P Q$) depending on three real parameters¹²; this will leave any W_P invariant. Thus we can very conveniently fix the radii of I_{P_1} and I_{P_2} to 1. At this point we must warn the reader that the relative sizes and arrangements of isometric circles are not projectively invariant, that is, they may change if a similarity transformation is made on all group elements. We will show however, that if P belongs to the group generated by P_1 and P_2 , then I_P and $I_{P^{-1}}$ are external (or externally tangent), so that $0 \leq W_P \leq 1$.

ξ is taken such that $\xi > \xi'$, as in the case when the transformation becomes multiplicative: $z' = W z$ ($\xi = \infty, \xi' = 0$). ξ being negative, the segments $[\xi_1', \xi_1]$, $[\xi_2', \xi_2]$ are exterior to one another. We can choose, for instance $\xi_1 > \xi_2$ so that we have the ordering $\xi_2' < \xi_2 < \xi_1' < \xi_1$. Thus we get the equations

$$\xi_1' = \xi_1 - \frac{1 - W_1}{(W_1)^{1/2}}, \quad \xi_2' = \xi_2 - \frac{1 - W_2}{(W_2)^{1/2}}, \quad (4.1)$$

$$\delta \xi = \xi_1 - \xi_2 > 0,$$

$$-\delta \xi^2 + \delta \xi \left(\frac{1 - W_1}{(W_1)^{1/2}} - \frac{1 - W_2}{(W_2)^{1/2}} \right) + \frac{(1 - W_1)(1 - W_2)}{(W_1 W_2)^{1/2}} \times (1 - \rho) = 0. \quad (4.2)$$

We are mostly interested in the isometric circles $I_{P_2}, I_{P_2^{-1}}, I_{P_1^{-1}}$, and I_{P_1} . Let us call their centers z_2', z_2, z_1' and z_1 . All these circles have unit radius. Then

$$z_1 - z_1' = (W_1)^{1/2} + (W_1)^{-1/2} \geq 2,$$

$$z_2 - z_2' = (W_2)^{1/2} + (W_2)^{-1/2} \geq 2, \quad (4.3)$$

$$\delta z = z_1' - z_2 = \delta \xi - (W_1)^{-1/2} - (W_2)^{1/2}$$

¹² It is convenient to use this technique of transforming the group by a similarity transformation because of property (d). The demonstration we give can also be done without applying this transformation simply by using the values of ξ_1, ξ_1', ξ_2 , and ξ_2' as given above in terms of x_1, x_2, W_1 , and W_2 . Making the radius of the isometric circle 1 is very convenient because only the relative position of the circles is important; however, it cannot be generalized to more than two loops.

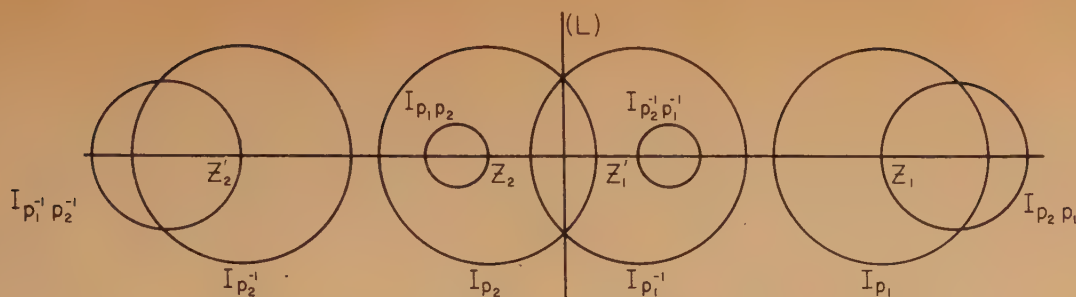


FIG. 2. Isometric circles associated with P_1 , P_2 , and P_1P_2 in the general case ($W_1W_2 \neq 1$, $z'_1 > z_2$).

satisfy the equation

$$(\delta z)^2 + \delta z \left(\frac{1+W_1}{W_1^{1/2}} + \frac{1+W_2}{W_2^{1/2}} \right) + \frac{(1-W_1)(1-W_2)}{(W_1W_2)^{1/2}} \left[\rho + \frac{(W_1^{1/2} + W_2^{1/2})^2}{(1-W_1)(1-W_2)} \right] = 0. \quad (4.4)$$

As we have chosen $\delta z > 0$, we must take the largest root of this equation. The inequality (3.18) ensures that (4.4) has two roots: one positive, one negative. Thus, the largest one satisfies $\delta z = z'_1 - z_2 \geq 0$. The limit $z'_1 = z_2$ is reached when $-\rho$ has its minimal value. Then the two circles I_1' and I_2 collapse into one single circle. Thus the inequalities

$$0 \leq W_1 \leq 1, \quad 0 \leq W_2 \leq 1, \quad -\rho \geq \frac{(W_1^{1/2} + W_2^{1/2})^2}{(1-W_1)(1-W_2)}$$

result in the following arrangement of the four isometric circles of unit radii:

$$z_1 - 2 \geq z'_1 \geq z_2 \geq z'_2 + 2. \quad (4.5)$$

Now we proceed to show that from this geometrical arrangement, we can rederive and extend the previous results.

Using Theorem 2, we see that z_2 lies in the circle $I_{P_1P_2}$, z'_1 in the circle $I_{P_2^{-1}P_1^{-1}}$. They have the arrangement indicated in Fig. 2. We see that $I_{P_1P_2}$ and $I_{P_2^{-1}P_1^{-1}}$ are thus exterior to each other, because $z'_1 \geq z_2$. In the limit $z'_1 = z_2$, they become tangent at z'_1 . Thus we find that $0 \leq W_{P_1P_2} \leq 1$ and that the limit $W_{P_1P_2} = 1$ can be reached (Fig. 3).

Let us show geometrically that $0 \leq W_{P_1^{-n}P_2^m} \leq 1$. The arrangement of the circles of the type $I_{P_1^n}$, $I_{P_2^m}$ is easy to study and is given in Fig. 4. We see that if we call L the perpendicular bisector of the segment $[z'_1, z_2]$, all the circles $I_{P_1^n}$ are on the right of L , except for $n = -1$. All the circles $I_{P_2^m}$ are on the left (except for $m = 1$). Using Theorem 3, we deduce that any circle $I_{P_1^nP_2^m}$ is contained in $I_{P_2^m}$ and is thus on the left of L (except for $n = -1$ or $m = -1$).

Any circle $I_{P_2^{-m}P_1^{-n}}$ is similarly contained in $I_{P_1^{-n}}$ and is on the right of L . As $I_{P_1^nP_2^m} \subset I_{P_2^m}$ and $I_{P_2^{-m}P_1^{-n}} \subset I_{P_1^{-n}}$, the circles $I_{P_2^m}$ and $I_{P_1^{-n}}$ have to be tangent if we are to have $W_{P_1^nP_2^m} = 1$. This is possible only for $m = 1$ and $n = 1$. We find again that only P_1P_2 can become parabolic.

Circles of the type $I_{P_1^{-1}P_2^m}$ and $I_{P_1^nP_2^{-1}}$ require a different treatment, because the circles $I_{P_2^{-m}} [m < 0]$ and $I_{P_1^{-1}}$ can intersect, and we cannot simply use Theorem 3. However, using Theorem 2, one can convince oneself that similarly $I_{P_1^{-1}P_2^m}$ is on the left of L , $I_{P_2^{-m}P_1^{-1}}$ on the right. Thus we conclude that these transformations are also hyperbolic. A special case occurs for $m = -1$. The radius of $I_{P_1^{-1}P_2^{-1}}$ is given by $r_{-1,-1} = 1/|z_2 - z'_1|$ and goes to infinity when $z'_1 = z_2$. The circles $I_{P_1^{-1}P_2^{-1}}$ and $I_{P_2P_1}$ are thus tangent at ∞ , and $W_{P_2P_1} = 1$. This is expected as $W_{P_2P_1} = W_{P_1P_2}$. We have thus rederived the fact that $W_{P_1^nP_2^m} < 1$ except for $n = m = 1$ (or $n = m = -1$). We have also shown that all the circles of the form $I_{P_1^nP_2^m}$ are on the left of L and all those of the form $I_{P_2^mP_1^n}$ are on the right.

It is now easy to extend these conditions to any transformation $\underline{L}$ of the type $P = P_1^n P_2^m P_1^q P_2^r$. As

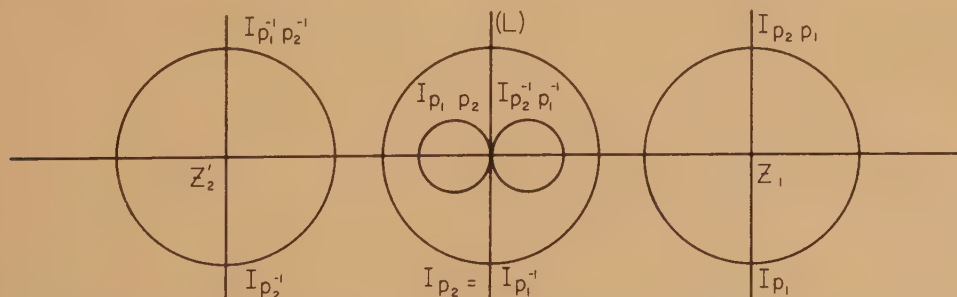


FIG. 3. Isometric circles associated with P_1P_2 , $(P_1P_2)^{-1}$, $(P_1^{-1}P_2^{-1})$, P_1 , and P_2 for $z'_1 = z_2$, showing that P_1P_2 and $P_1^{-1}P_2^{-1}$ degenerate into parabolic transformations.

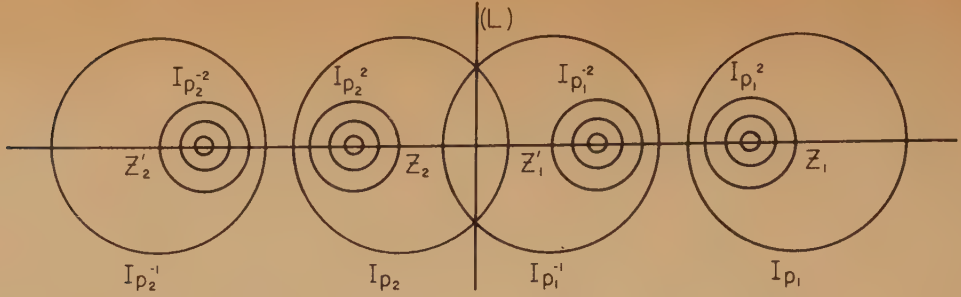


FIG. 4. Isometric circles associated with P_1^n and P_2^m in the general case.

$I_{P_2^{-1}P_1^{-1}}$ is on the right of L and $I_{P_1^n P_2^m}$ on the left, we use Theorem 3 to deduce that I_P is included in $I_{P_1^n P_2^m}$ and thus on the left of L . Likewise, we find that $I_{P^{-1}}$ is included in $I_{P_2^{-m}P_1^{-n}}$ and is therefore on the right of L . We can conclude that $W_P \leq 1$. For W_P to be 1, we must have the circles $I_{P_1^n P_2^m}$ and $I_{P_2^{-m}P_1^{-n}}$ tangent on L as well as the circles $I_{P_1^n P_2^m}$ and $I_{P_2^{-m}P_1^{-n}}$ tangent on L . (The first condition is necessary if we are to have I_P and $I_{P^{-1}}$ tangent on L ; the second condition is guaranteed by the second part of Theorem 3.) Both conditions are required for the equality $W_P = 1$. Since all four circles are tangent on L , the circles $I_{P_1^n P_2^m}$ and $I_{P_2^{-m}P_1^{-n}}$ are tangent, which means that $r = q = 1$ or $r = q = -1$. Likewise, the circles $I_{P_1^n P_2^m}$ and $I_{P_2^{-m}P_1^{-n}}$ are tangent, meaning that $n = m = 1$ or $n = m = -1$. We are now left with four possible cases: $n = m = 1$ and $r = q = 1$, $n = m = -1$ and $r = q = -1$, $n = m = 1$ and $r = q = -1$, and $n = m = -1$ and $r = q = 1$. By closely examining Figs. 2-4, we see that only the first two conditions allow I_P and $I_{P^{-1}}$ to become tangent. (This is easy to see. Notice that the circles for $P_1 P_2 P_1 P_2$ and $P_2^{-1} P_1^{-1} \times P_2^{-1} P_1^{-1}$ can become tangent on L . The circles for $P_1^{-1} P_2^{-1} P_1^{-1} P_2^{-1}$ and $P_2 P_1 P_2 P_1$ can become parallel lines, in which case they become tangent to L at infinity. The other circles, however, lie within other circles which never become tangent, thereby excluding the other two cases.) The result easily generalizes for arbitrary elements of P , so the only divergent factors are associated with P_1^n , P_2^n , and $(P_1 P_2)^n$. Remarkably, all self-crossing closed paths of the dual diagram do not contribute divergent factors. [Bear in mind that factors like $(P_1 P_2)^{-n}$ are not included because of (c).]

Thus, in the two-loop case, three transformations are playing a symmetric role: P_1 , P_2 , and $P_1 P_2$. This is expected, since if we take any two of them we can generate the whole group generated by P_1 and P_2 .

V. CONCLUSION

We have shown that the two-loop tadpole diverges faster than the product of two divergences. The additional divergence is of the same exponential type as the previous ones and it occurs also at an endpoint of the integration region. The divergent factor can be written

$$\left\{ \prod_{n=1}^{\infty} (1 - W_1^n)(1 - W_2^n)[1 - (W_{P_1 P_2})^n] \right\}^{-4}.$$

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APPENDIX A: BOUND ON CROSS RATIO ζ

Let us define f by the equation

$$\zeta = -f / (1 - W_1)(1 - W_2),$$

where

$$f = \frac{[1 - x_1(1 - x_2 W_1)][(1 - x_1)(1 - x_2) + x_2 W_2]}{x_1 x_2 (1 - x_1)(1 - x_2)} \geq 0. \quad (A1)$$

$f(x_1, x_2)$ has thus a minimum inside the square $0 \leq x_1, x_2 \leq 1$. This minimum can be found by taking partial derivatives,

$$\frac{\partial f}{\partial x_1} = 0 \Leftrightarrow \frac{1}{(1 - x_1)^2} = \frac{1 - x_2(1 - W_2)}{W_1 W_2 x_2^2 x_1^2}, \quad (A2)$$

$$\frac{\partial f}{\partial x_2} = 0 \Leftrightarrow \frac{1}{(1 - x_2)^2} = \frac{(1 - x_1)^2}{W_2 x_2^2 [1 - x_2(1 - W_1)]}. \quad (A3)$$

f can be rewritten:

$$f = W_1 + W_2 + \frac{1 - x_1}{x_1 x_2} + W_2 \frac{(1 - x_1)}{x_1} + W_2 x_2 \frac{1 - x_1(1 - W_1)}{x_1(1 - x_1)(1 - x_2)}.$$

Using the values of x_1 and x_2 given by (A2) and (A3), we find that

$$f \geq W_1 + W_2 + \frac{W_2(1 - x_1)}{x_1} + (W_1 W_2)^{1/2} [1 - x_2(1 - W_2)]^{-1/2} + W_2^{1/2} [1 - x_1(1 - W_1)]^{1/2} x_1^{-1},$$

and finally

$$f \geq W_1 + W_2 + 2(W_1 W_2)^{1/2}. \quad (A4)$$

We show now that $f = W_1 + W_2 + 2(W_1 W_2)^{1/2}$ can be

satisfied. Equations (A2) and (A3) have a common root, which is

$$1 - x_1 = x_2 = 0.$$

Let us go to this limit with $(1 - x_1)/x_2 \rightarrow \zeta$. Then

$$f = (\zeta + W_1)(\zeta + W_2)/\zeta = \zeta + (W_1 + W_2) + W_1 W_2/\zeta.$$

But

$$\frac{\partial f}{\partial \zeta} = 1 - \frac{W_1 W_2}{\zeta^2},$$

so therefore

$$f \text{ is minimal for } \zeta = (W_1 W_2)^{1/2}. \quad (\text{A5})$$

For this value,

$$f = (W_1^{1/2} + W_2^{1/2})^2 = W_1 + W_2 + 2(W_1 W_2)^{1/2}.$$

Thus

$$-\zeta \geq (W_1^{1/2} + W_2^{1/2})^2 / (1 - W_1)(1 - W_2). \quad (\text{A6})$$

The equality can be reached for $(1 - x_1)/x_2 = (W_1 W_2)^{1/2}$, $1 - x_1 = 0$, and $x_2 = 0$.

APPENDIX B

The calculations which have been performed by sewing tadpoles can be confirmed by using the Koba-Nielsen variables originally used to compute the N -loop graph. However, this time the cross ratio ζ is a function of six variables and this complicates the search for the minimum value (see Fig. 5).

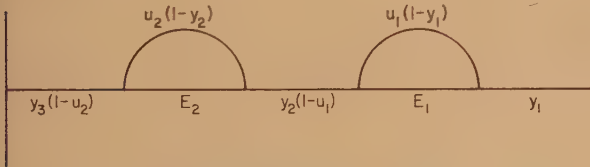


FIG. 5. Two-loop graph with projective variables, taken from the original N -loop calculation.

We shall only consider the obvious minima found by evaluating various partial derivatives. In general, however, other minima can conceivably exist on the faces of the hypercube.

In Ref. 4, the projective operators are given by

$$P_1 = y_1 \begin{pmatrix} 1 & -E_1 \\ 1 & -E_1(1 - \bar{u}_1) \end{pmatrix} y_1^{-1},$$

$$P_2 = y_1 y_2 E_1 (1 - u_1) \begin{pmatrix} 1 & -E_2 \\ 1 & -E_2(1 - \bar{u}_2) \end{pmatrix} \times [y_1 y_2 E_1 (1 - u_1)]^{-1}, \quad (\text{B1})$$

where P_1 and P_2 are expressed as matrices. We define the various variables as follows ($0 \leq y_1, y_2, t_1, t_2, u_1, u_2 \leq 1$):

$$E_1 = \frac{t_1(1 - t_1)}{1 - t_1[1 - u_1(1 - y_1)]},$$

$$E_2 = \frac{t_2(1 - t_2)}{1 - t_2[1 - u_2(1 - y_2)]}, \quad (\text{B2})$$

$$\bar{u}_1 = u_1(1 - y_1), \quad \bar{u}_2 = u_2(1 - y_2).$$

The invariant points of P_1 and P_2 are given by

$$\zeta_1 = y_1 E_1 t_1^{-1}, \quad \zeta_1' = y_1 t_1,$$

$$\zeta_2 = y_1 y_2 E_1 (1 - u_1) E_2 t_2^{-1}, \quad \zeta_2' = y_1 y_2 E_1 (1 - u_1) t_2, \quad (\text{B3})$$

$$0 \leq \zeta_2' \leq \zeta_2 \leq \zeta_1' \leq \zeta_1 \leq 1.$$

The multipliers are defined as follows:

$$W_1 = W_{P_1} = \bar{u}_1 t_1 / (1 - t_1) [1 - t_1(1 - \bar{u}_1)] \quad (0 \leq W_1 \leq 1),$$

$$W_2 = W_{P_1} = \bar{u}_2 t_2 / (1 - t_2) [1 - t_2(1 - \bar{u}_2)] \quad (0 \leq W_2 \leq 1). \quad (\text{B4})$$

It is not hard to show that

$$u_{11}^{-1} = \frac{\text{Tr}[P_1 P_2]}{(\det P_1 P_2)^{1/2}} = \frac{1 - y_2^{-1}(1 - u_1)^{-1} + E_1 E_2 [(1 - \bar{u}_1)(1 - \bar{u}_2) - y_2(1 - u_1)]}{(\bar{u}_1 \bar{u}_2 E_1 E_2)^{1/2}}. \quad (\text{B5})$$

We wish to minimize the function $|u_{11}^{-1}|$. Let

$$a = 1 - y_2^{-1}(1 - u_1)^{-1}, \quad b = (1 - \bar{u}_1)(1 - \bar{u}_2) - y_2(1 - u_1),$$

and

$$c = \bar{u}_1 \bar{u}_2. \quad (\text{B6})$$

Taking a partial derivative with respect to t_1 , we find that

$$\partial u_{11}^{-1} / \partial t_1 = (2c E_1 E_2)^{-1/2} (-a/E_1 + E_2 b) \times \partial E_1 / \partial t_1 = 0. \quad (\text{B7})$$

Thus u_{11}^{-1} has an extremum with respect to t_1 either for

$$(I) -a + E_1 E_2 b = 0 \quad \text{or} \quad (II) \frac{\partial E_1}{\partial t_1} = 0. \quad (\text{B8})$$

Let us first study case (I):

$$E_1 E_2 \geq 0, \quad a \leq 0, \quad b = (1 - u_1)(1 - u_2)(1 - y_2) + u_1 y_1 (1 - \bar{u}_2). \quad (\text{B9})$$

We see that (I) implies $a = 0$ and $E_1 E_2 b = 0$. $a = 0$ implies $y_2 = 1$ and $u_1 = 0$. Therefore, we find that $b = 0$.

Let us study u_{11}^{-1} in the limit

$$1-y_2 \rightarrow 0, \quad u_1 \rightarrow 0, \quad (1-y_2)/u_1 \rightarrow \eta \geq 0. \quad (\text{B10})$$

We see that

$$\begin{aligned} a &\sim -u_1(1+\eta), \\ b &\sim u_1[y_1 + \eta(1-u_2)], \\ E_1 &\sim t_1, \quad E_2 \sim t_2, \end{aligned} \quad (\text{B11})$$

so that

$$u_{11}^{-1} = \frac{-(1+\eta) + t_1 t_2 [y_1 + \eta(1-u_2)]}{[(1-y_1)u_2 \eta t_1 t_2]^{1/2}}. \quad (\text{B12})$$

We see that u_{11}^{-1} is an increasing function with respect to $t_1 t_2$; when $t_1 t_2 = 0$, $u_{11}^{-1} = -\infty$. Thus u_{11}^{-1} is maximal for $t_1 t_2 = 1$, in which case

$$u_{11}^{-1} = (-1 + y_1 - \eta u_2) / [(1-y_1)\eta u_2]^{1/2}. \quad (\text{B13})$$

If we let $\lambda = 1-y_1$ and $k = \eta u_2$, then

$$u_{11}^{-1} = -(\lambda + k) / (k\lambda)^{1/2} \leq -2. \quad (\text{B14})$$

Thus we have proved that for this minimum, $|u_{11}^{-1}| \geq 2$. The equality $|u_{11}^{-1}| = 2$ is reached for

$$\begin{aligned} 1-y_2 &\sim \eta u_1 \rightarrow 0, \\ t_1 &= t_2 = 1, \\ 1-y_1 &= \eta u_2. \end{aligned} \quad (\text{B15})$$

One sees that to maintain the constraint on the integration region $0 \leq W_{P_1} \leq 1$, $0 \leq W_{P_2} \leq 1$, one needs also $u_2 \rightarrow 0$ and $y_1 \rightarrow 1$. Thus, in the limit

$$\begin{aligned} 1-y_2 &\sim \eta u_1 \rightarrow 0, \\ 1-t_1 &= 1-t_2 = 0, \\ 1-y_1 &\sim \eta u_2 \rightarrow 0, \end{aligned} \quad (\text{B16})$$

$|u_{11}^{-1}|$ reaches 2, so $W_{P_1 P_2} = 1$. In this limit, W_{P_1} and W_{P_2} are indeterminate and can have any value between zero and 1:

$$W_{P_1} = \lim_{u_1 \rightarrow 0, 1-y_1 \rightarrow 0, 1-t_1 \rightarrow 0} \left[\frac{u_1(1-y_1)}{(1-t_1)^2} \right], \quad (\text{B17})$$

$$W_{P_2} = \lim \left[\frac{u_2(1-y_2)}{(1-t_2)^2} \right].$$

Let us now examine solution (II):

$$\begin{aligned} \frac{\partial E_1}{\partial t_1} &= 0, \\ \frac{\partial E_1}{\partial t_1} &= \frac{1-2t_1+t_1^2(1-\bar{u}_1)}{[1-t_1(1-\bar{u}_1)]^2} = \frac{(\zeta_1 - \zeta_1')}{y_1[1-t_1(1-\bar{u}_1)]}. \end{aligned} \quad (\text{B18})$$

Thus $\partial E_1 / \partial t_1 = 0$ implies $\zeta_1 = \zeta_1'$ and $W_{P_1} = 1$. Also

$$\frac{\partial E_2}{\partial t_2} = \frac{(\zeta_2 - \zeta_2')}{y_1^2 y_2 (1-u_1) E_1 [1-t_2(1-\bar{u}_2)]} = 0 \quad (\text{B19})$$

implies $W_{P_2} = W_2 = 1$.

In this limit, it is easier to minimize the cross ratio ζ of the four invariant points rather than $|u_{11}^{-1}|$. We will study f , where

$$\begin{aligned} f &= -\zeta(1-W_1)(1-W_2) \\ &= -\frac{(\zeta_1' - \zeta_2)(\zeta_2' - \zeta_1)}{(\zeta_1' - \zeta_1)(\zeta_2' - \zeta_2)} (1-W_1)(1-W_2), \end{aligned} \quad (\text{B20})$$

in the limit $\zeta_1' = \zeta_1$, $\zeta_2' = \zeta_2$. Using the following facts,

$$\begin{aligned} \zeta_1 - \zeta_1' &= y_1(1-t_1)(1-W_1), \\ \zeta_2 - \zeta_2' &= y_1 y_2 E_1(1-u_1)(1-t_2)(1-W_2), \end{aligned} \quad (\text{B21})$$

we can show that

$$\begin{aligned} f &= + \frac{(\zeta_1 - \zeta_2)^2}{y_1^2 y_2 E_1(1-u_1)(1-t_1)(1-t_2)} \\ &= \frac{[1-y_2 t_1 t_2(1-u_1)]^2}{y_2(1-u_1)(1-t_1)(1-t_2)}. \end{aligned} \quad (\text{B22})$$

Let $\nu = y_2(1-u_1)$. Then we have

$$\frac{\partial f}{\partial \nu} = - \frac{1}{\nu^2} \frac{(1-t_1 t_2 \nu)(\nu t_1 t_2 + 1)}{(1-t_1)(1-t_2)} < 0. \quad (\text{B23})$$

f is thus a decreasing function with respect to ν . We see, therefore, that

$$f \geq (1-t_1 t_2)^2 / (1-t_1)(1-t_2). \quad (\text{B24})$$

The equality requires the limit $y_2 \rightarrow 1$ and $u_1 \rightarrow 0$. But the condition $W_1 = W_2 = 1$ implies that the limits $t_1 \rightarrow 1$, $t_2 \rightarrow 1$, $y_1 \rightarrow 1$, $u_2 \rightarrow 0$ are also taken, as before.

In the limit $t_1 \rightarrow 1$, $t_2 \rightarrow 1$, $1-t_1 = \epsilon_1$, $1-t_2 = \epsilon_2$, we find

$$f \geq (\epsilon_1 + \epsilon_2)^2 / \epsilon_1 \epsilon_2 \geq 4. \quad (\text{B25})$$

$f = 4$ is reached only for $\epsilon_1 = \epsilon_2$. Thus, the second solution is a special case of the first one:

$$\begin{aligned} y_1, y_2 &\rightarrow 1, & \bar{u}_1(1-t_1)^{-2} &\rightarrow 1, \\ u_1, u_2 &\rightarrow 0, & \bar{u}_2(1-t_2)^{-2} &\rightarrow 1, \\ t_1, t_2 &\rightarrow 1, & (1-t_1)/(1-t_2) &\rightarrow 1. \end{aligned} \quad (\text{B26})$$

As before, the limits $f = 4$ and $W_1 = W_2 = 1$ imply $W_{P_1 P_2} = 1$.

In conclusion, we see that the evaluation of the minimum gives $W_{P_1 P_2} \leq 1$. The equality is reached in some complicated limit, not necessarily for $W_1 = W_2 = 1$. Other minima could in principle exist on the faces of the hypercube when approached at different angles.

Divergence of the N -Loop Planar Graph in the Dual-Resonance Model*

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Many of the results of a previous paper on the divergence of the two-loop planar graph are generalized to the multiloop case. In particular, the N -loop amplitude is shown to diverge in at least $2^N - 1$ places, though no attempt is made to show that these are the only divergences. The implication of this result to the renormalization program is discussed.

I. INTRODUCTION

NOW that the connection between the multiloop dual-resonance amplitudes^{1–3} and the analog model^{2,3} has been made, we find compelling reasons to

believe that the singularities of the N -loop integrand are associated with a change of the topology of the corresponding Riemann surface. In the single-loop case, for example, the associated surface is a disk with a hole; the singularity of the partition function arises when the hole shrinks to a point. Likewise, we predict that the

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¹ M. Kaku and L. P. Yu, LRL Report Nos. UCRL-20054,

UCRL-20055, UCRL-20104, 1970 (unpublished); Phys. Letters **33B**, 166 (1970).

² C. Lovelace, Phys. Letters **32B**, 703 (1970).

³ V. Alessandrini, CERN Report No. TH. 1215, 1970 (unpublished).

double-loop amplitude contains three singularities, two associated with the individual holes shrinking to points, and one associated with the two circles touching each other. In general, we expect $2^N - 1$ divergent factors in the N -loop case, corresponding to the shrinking of the individual circles and the various ways in which these circles can touch.

Though this picture gives a simple and intuitive approach to the problem of isolating the divergences of the multiloop amplitude, the actual problem is quite tediously involved, because of the existence of an infinite number of potential singularities, each associated with a closed path of the Kikkawa-Sakita-Virasoro⁴ (KSV) dual diagram.

In a previous paper,⁵ we rigorously proved that the double-loop amplitude diverged in only three places, corresponding to the set of closed paths which did not cross themselves. In this paper, we make no attempt to prove that all self-crossing lines do not diverge because the method of isometric circles, which was employed to get advantage in the double-loop case, fails for the arbitrary case. Because only two projective transformations can have isometric circles with unit radii at any given time, we can only generalize our results to the paths which encircle only two internal points of the dual diagram.

In Sec. II, we express the singularity in terms of projectively invariant cross ratios of invariant points. Particular emphasis is given to the complicated interrelations among these cross ratios. In Sec. III, we actually construct the $2^N - 1$ singularities found in the N -loop planar graph.

The cataloging of these divergences is critical to the renormalization program.⁶ It has been shown that the additional divergence appearing in the double-loop case can be renormalized by introducing appropriate

counterterms similar to the one defined in the single-loop case.⁷

II. MULTILoop SINGULARITY

In our previous paper,⁵ we showed that it is possible to express the multiplier of the product of two projective transformations in terms of the individual multipliers and invariant or fixed points. We found that if two transformations P_1 and P_2 have multipliers given by W_1 and W_2 and invariant points given by (x_1, x_1') and (x_2, x_2') , respectively, then the multiplier of the product $(W_{P_1 P_2})$ is given by

$$W_{P_1 P_2}^{1/2} + W_{P_1 P_2}^{-1/2} = \text{Tr}(P_1 P_2) [\det(P_1 P_2)]^{-1/2} \\ = |W_1 + W_2 + \rho_{12}(1 - W_1)(1 - W_2)| \\ \times (W_1 W_2)^{-1/2}, \quad (2.1)$$

where

$$\rho_{12} = \frac{(x_1 - x_2')(x_2 - x_1')}{(x_1 - x_1')(x_2 - x_2')}.$$

The multiloop singularity, unfortunately, involves multipliers of large numbers of projective transformations; in fact, the singularity includes an arbitrarily large number of products, one for each closed path of the dual diagram:

$$\prod_{\{R\}} (1 - W_R)^{-4}, \quad (2.2)$$

where $\{R\}$ is the group generated by N projective transformations P_i and their inverses, such that cyclic permutations and inverses of such permutations of previously included members of $\{R\}$ are omitted. Our first task, then, is to generalize (2.1) for the arbitrary case. Once the multiplier of the product of three projective transformations is analyzed in detail it is not hard to generalize to the general case:

$$W_{P_1^{\alpha(1)} P_2^{\alpha(2)} \dots P_N^{\alpha(N)} 1/2} + W_{P_1^{\alpha(1)} P_2^{\alpha(2)} \dots P_N^{\alpha(N)} -1/2} = \frac{\text{Tr}(P_1^{\alpha(1)} P_2^{\alpha(2)} \dots P_N^{\alpha(N)})}{[\det(P_1^{\alpha(1)} P_2^{\alpha(2)} \dots P_N^{\alpha(N)})]^{1/2}} \\ = \left| 2 - \sum_{i=1}^N (1 - W_i^{\alpha(i)}) + \sum_{i>j}^N \rho_{ij} (1 - W_i^{\alpha(i)}) (1 - W_j^{\alpha(j)}) \right. \\ \left. - \sum_{i>j>k}^N \rho_{ijk} (1 - W_i^{\alpha(i)}) (1 - W_j^{\alpha(j)}) (1 - W_k^{\alpha(k)}) + \dots + (-1)^N \rho_{123\dots N} \prod_{i=1}^N (1 - W_i^{\alpha(i)}) \right| \left(\prod_{i=1}^N W_i^{\alpha(i)} \right)^{-1/2}, \quad (2.3)$$

where

$$\rho_{ij\dots m} = \frac{(x_i - x_j')(x_j - x_k') \dots (x_m - x_i')}{(x_i - x_i')(x_j - x_j') \dots (x_m - x_m')}, \quad \alpha(i) \text{ are integers.}$$

⁴ K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969).

⁵ M. Kaku and J. Scherk, Phys. Rev. D **3**, 430 (1971).

⁶ D. J. Gross, A. Neveu, J. Scherk, and J. H. Schwarz, Phys. Rev. D **2**, 697 (1970).

⁷ E. Cremmer and A. Neveu, Orsay report, 1970 (unpublished).

Our task, then, is to find solutions of the equation

$$W_{P_1 P_2 \dots P_N} = 1, \quad (2.4)$$

subject to the conditions implied by (2.3).

Since the proof of (2.3) proceeds easily by induction, we will sketch the proof. Because we know that (2.1) is valid, we will take $(P_1 P_2 \dots P_{N-1})$ and P_N as our two projective transformations:

$$\begin{aligned} & W_{(P_1 P_2 \dots P_{N-1}) P_N}^{1/2} + W_{(P_1 P_2 \dots P_{N-1}) P_N}^{-1/2} \\ &= [W_{12 \dots N-1} + W_N + \rho_{(12 \dots N-1)N} (1 - W_N) \\ &\quad \times (1 - W_{(12 \dots N-1)})] (W_N W_{(12 \dots N-1)})^{-1/2} \\ &\equiv D_N (W_N W_{(12 \dots N-1)})^{-1/2}. \end{aligned} \quad (2.5)$$

We see immediately that D_N is linear in W_N . But we could equally well have chosen P_1 and $(P_2 P_3 \dots P_N)$ as our two projective transformations, or any other cyclic permutation. Therefore, D_N must be linear in all W_i . We can always choose D_N to be of the form

$$\begin{aligned} D_N &= A + \sum_{i=1}^N \alpha_i W_i + \sum_{i>j}^N \sigma_{ij} (1 - W_i) (1 - W_j) \\ &\quad - \sum_{i>j>k}^N \sigma_{ijk} (1 - W_i) (1 - W_j) (1 - W_k) + \dots \\ &\quad + (-1)^N \sigma_{12 \dots N} \prod_{i=1}^N (1 - W_i), \end{aligned} \quad (2.6)$$

where A , α_i , and the σ 's are unspecified functions of the invariant points x_i and x_i' . We must prove the equivalence of these functions with the ρ 's. Assume now that (2.6) is valid for all orders less than or equal to $N-1$. In the limit $W_i \rightarrow 1$, D_N (which is as yet undetermined) must reduce to D_{N-1} . It is easy to see that D_N reduces to D_{N-1} if we can equate the A , α_i , and σ 's to the corresponding ρ 's. The only undetermined function is $\sigma_{12 \dots N}$, because this factor vanishes whenever one of the W_i is set equal to 1. We obtain conditions on this last undetermined function if we let $x_N = x_{N-1}$ and $x_N' = x_{N-1}'$ (i.e., $W_{N,N-1} = W_N W_{N-1}$). In this limit, we have

$$D_N(W_1, W_2, \dots, W_N) \rightarrow D_{N-1}(W_1, W_2, \dots, W_{N-1} W_N),$$

so therefore

$$\sigma_{12 \dots N}(x_N = x_{N-1}, x_N' = x_{N-1}') \rightarrow \sigma_{12 \dots N-1}.$$

Likewise, in the case $x_N = x_{N-1}'$ and $x_N' = x_{N-1}$ (i.e., $W_{N-1,N} = W_{N-1} W_N^{-1}$) we have

$$D_N(W_1, W_2, \dots, W_N) \rightarrow D_{N-1}(W_1, W_2, \dots, W_{N-1} W_N^{-1})$$

and therefore

$$\sigma_{12 \dots N}(x_N = x_{N-1}', x_N' = x_{N-1}) \rightarrow 0.$$

There is only one function that satisfies these two conditions along with cyclic symmetry and certain linearity conditions imposed by (2.5), and this function is the cross ratio. Q.E.D.

Now that (2.3) is proven we look for solutions of (2.4). Unfortunately, the problem is complicated because of certain interrelations that exist among ρ 's of the same order and ρ 's of different orders. Our goal is to express (2.3) entirely in terms of a fundamental set of variables, say, the set of all second-order ρ 's.

We shall list some of these relations. It is not hard to show that both ρ_{123} and ρ_{321} are solutions of the quadratic equation

$$k^2 - k(\rho_{12} + \rho_{23} + \rho_{31} - 1) + \rho_{12} \rho_{23} \rho_{31} = 0 \quad (2.7)$$

and that ρ_{1234} and ρ_{4321} are solutions of

$$\begin{aligned} k^2 - k(\rho_{13} \rho_{34} + \rho_{23} \rho_{41} - \rho_{13} \rho_{24} + \rho_{24} + \rho_{13} - 1) \\ + \rho_{12} \rho_{23} \rho_{34} \rho_{41} = 0. \end{aligned} \quad (2.8)$$

Also, we have the condition

$$\rho_{ij \dots mn} \rho_{nm \dots ji} = \rho_{ij} \rho_{jk} \dots \rho_{mn} \rho_{ni}. \quad (2.9)$$

To further complicate the situation, it can be shown that the second-order cross ratios obey the KSV duality conditions. For example, let $\rho_{i,i+1}$ form the sides of an N -sided polygon, while the other $\rho_{i,j}$ form the various diagonals. The relationship between crossed lines of the polygon form the usual KSV relations. If we let $x_1 = 0$, $x_1' = \infty$, and $x_2 = 1$, we can explicitly calculate all ρ 's of second (and higher) order in terms of a fundamental set of second-order ρ 's.

Fortunately, there are two important reasons why our goal is feasible.

(1) If we take the limit $W_i \rightarrow 1$ (i.e., $x_i \rightarrow x_i'$),⁸ then all higher-order ρ 's reduce to simple products of second-order ρ 's. For example, multiply both sides of (2.9) by products of $1 - W_i$ and then take the limit as all W_i 's tend to 1:

$$\begin{aligned} \lim_{W \rightarrow 1} \rho_{i_1 i_2 \dots i_N} \prod_{j=1}^N (1 - W_j) \lim_{W \rightarrow 1} \rho_{i_N i_{N-1} \dots i_1} \prod_{k=1}^N (1 - W_k) \\ = (-1)^N \bar{\rho}_{i_1 i_2 \dots i_N}^2 = \bar{\rho}_{i_1 i_2} \bar{\rho}_{i_2 i_3} \dots \bar{\rho}_{i_N i_1}, \end{aligned} \quad (2.10)$$

where

$$\bar{\rho}_{i_1 i_2 \dots i_N} = \lim_{W \rightarrow 1} \rho_{i_1 i_2 \dots i_N} \prod_{j=1}^N (1 - W_j).$$

In the limit $W_i \rightarrow 1$, we see that our problem is reduced to evaluating all second-order $\bar{\rho}$'s.

(2) The second reason why our task simplifies is because higher-order ρ 's decompose into products of lesser ρ 's if any of the indices are repeated:

$$\rho_{ijk \dots mnj pqr} = \rho_{ij} \rho_{pq} \rho_{kl} \dots \rho_{mnj}. \quad (2.11)$$

⁸ If linear dependences are neglected, then the projective transformations reduce to the identity in the limit where the multiplier goes to 1. Because of linear dependences, however, the invariant points become equal in the limit where the multiplier goes to 1, in which case the transformation is said to be parabolic. Notice that in (2.3), if all multipliers are set to 1, then we always have a solution for (2.4), meaning that all closed paths of the dual diagram diverge in the case where linear dependences are ignored.

Since self-crossing paths involve repeating one or more of the N projective transformations in the singularity (2.2), we will be able to reduce the complexity of the problem once we know (2.11). In the triple-loop case, for example, a knowledge of ρ_{12} , ρ_{23} , ρ_{31} , ρ_{123} , and ρ_{321} completely determines the nature of all paths.

III. CONSTRUCTION OF SOLUTIONS OF (2.4)

Now that all the mathematical preliminaries have been presented, we will explicitly display solutions of (2.4) in terms of the original N -loop variables. We will refer extensively to the notation of our previous paper.

In that paper, we showed that $W_{12} \rightarrow 1$ independently of the specific values of W_1 and W_2 , but that the equations simplified enormously if we took the limit W_1 and $W_2 \rightarrow 1$. We will take the limit $W_i \rightarrow 1$ in order to take advantage of (2.10).

As before, we find that (see Fig. 1)

$$P_i = z_i \begin{pmatrix} 1 & -E_i \\ 1 & -E_i(1-\bar{u}_i) \end{pmatrix} z_i^{-1},$$

$$x_i = \text{invariant point} = P_i(x_i) = z_i E_i t_i^{-1},$$

$$x'_i = \text{invariant point} = P_i(x'_i) = z_i t_i,$$

$$E_i = t_i(1-t_i) [1-t_i(1-\bar{u}_i)],$$

$$\bar{u}_i = u_i(1-y_i),$$

$$z_i = y_i \prod_{j=1}^{i-1} y_j E_j (1-u_j),$$

$$W_i = \text{multiplier} = \bar{u}_i t_i (1-t_i)^{-1} [1-t_i(1-\bar{u}_i)]^{-1},$$

$$0 \leq y_i, u_i, t_i \leq 1, \quad 0 \leq W_i \leq 1$$

and

$$x_1 \geq x'_1 \geq x_2 \geq x'_2 \geq \dots \geq x_N' . \quad (3.1)$$

In the limit $W_i \rightarrow 1$, we find that

$$\bar{\rho}_{ij} = \lim_{W \rightarrow 1} \rho_{ij} (1-W_i)(1-W_j) = - \frac{(1-v_{ij} t_i t_j)^2}{(1-t_i)(1-t_j) v_{ij}}, \quad (3.2)$$

where

$$v_{ij} = z_i z_j^{-1} E_j^{-1} \quad (i > j). \quad (3.3)$$

In the double-loop case, we have a solution of (2.4) if we let $v_{12} \rightarrow 1$; then $t_1 = t_2 \rightarrow 1$ (i.e., $\bar{\rho}_{12} = -4$). In the triple-loop case, we must perform a more complicated limiting procedure:

$$t_1 = t_2 = t_3 \cong E_1^{1/2} = E_2^{1/2} = E_3^{1/2} \cong 1 - \bar{u}_1^{1/2} = 1 - \bar{u}_2^{1/2} \\ = 1 - \bar{u}_3^{1/2} = 1 - u_2 = y_2 \rightarrow 1, \quad (3.4)$$

$$1 - y_2 \gg u_1, \quad 1 - y_3 \ll u_2.$$

In this limit, we find that $\bar{\rho}_{12} = -9$, $\bar{\rho}_{23} = -9$, and $\bar{\rho}_{31} = -36$.

It is not hard to generalize from the triple-loop case

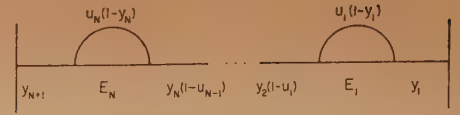


FIG. 1. Variables found in the multiloop calculation.

to the general case:

$$t_i = t_{i+1} \cong E_i^{1/2} = E_{i+1}^{1/2} \cong 1 - \bar{u}_i^{1/2} = 1 - \bar{u}_{i+1}^{1/2} \cong y_j \\ = y_{j+1} = 1 - u_k = 1 - u_{k+1} \rightarrow 1 \quad (3.5)$$

for $i=1, \dots, N-1$, $j=2, \dots, N-2$, $k=2, \dots, N-1$ and $1-y_2 \gg u_1$ and $1-y_N \ll u_{N-1}$.

With this particular choice of limits, we easily find

$$\bar{\rho}_{ij} = -[4(i-j)]^2, \quad i, j \neq 1 \text{ or } N \\ \bar{\rho}_{Ni} = -[4(N-i)-1]^2, \quad i \neq 1 \\ \bar{\rho}_{i1} = -[4(i-1)-1]^2, \quad i \neq N \\ \bar{\rho}_{N1} = -[4(N-1)-2]^2. \quad (3.6)$$

The proof that (3.6) satisfies (2.4) is long and tedious so only the highlights of the proof will be presented. In general, we wish to perform summations like the following:

$$\lim_{W \rightarrow 1} \sum_{i > j > k=2}^{N-1} \rho_{ijk} (1-W_i)(1-W_j)(1-W_k) \\ = \sum_{i > j > k=2}^{N-1} -(\bar{\rho}_{ij} \bar{\rho}_{jk} \bar{\rho}_{ki})^{1/2} \\ = \sum_{i > j > k=2}^{N-1} 4^3 (i-j)(j-k)(k-i). \quad (3.7)$$

Clearly, our task is to find a closed expression for

$$\sum_{i_1 > i_2 > \dots > i_n=2}^{N-1} (i_1-i_2)(i_2-i_3) \dots (i_{n-1}-i_n)(i_n-i_1). \quad (3.8)$$

As a first step, it is not hard to show that

$$f_1(p) \equiv \sum_{j=k+1}^{i-1} (i-j)(j-k) = \frac{(p+1)p(p-1)}{2 \times 3},$$

$$p = i - k. \quad (3.9)$$

If we define

$$f_n(i_1 - i_{n+2}) \\ \equiv \sum_{\substack{i_2 > i_3 > \dots > i_{n+1} \\ (i_1 > i_2, i_{n+1} > i_{n+2})}} (i_1-i_2)(i_2-i_3) \dots (i_{n+1}-i_{n+2}) \quad (3.10)$$

and

$$g_n(p) \equiv \binom{p+n}{2n+1}, \quad (3.11)$$

then we suspect that $f_n = g_n$ for all n . The proof of the last statement is easy once we establish two more

identities:

$$\sum_{p=n+1}^N g_n(p) = g_{n+1}(N+1)(2n+3)(N+n+2)^{-1} \quad (3.12)$$

and

$$\sum_{p=n+1}^N p g_n(p) = g_{n+1}(N) \times (2N-1)(n+1)(N+n+1)^{-1}. \quad (3.13)$$

(These last two statements are easily proven by induction.)

We can now prove that $f_n = g_n$ by induction. Assume that the statement holds for all orders less than or equal to n . Then we have

$$\begin{aligned} f_{n+1}(i-k) &= \sum_{j=k+n+1}^{i-1} (i-j) f_n(j-k) = \sum_{p=n+1}^{i-k-1} (i-k-p) g_n(p) \\ &= (i-k) g_{n+1}(i-k) (2n+3) (i-k+n+1)^{-1} \\ &\quad - g_{n+1}(i-k) (2n+2) (i-k-\frac{1}{2}) (i-k+n+1)^{-1} \quad (3.14) \\ &= g_{n+1}(i-k). \quad \text{Q.E.D.} \end{aligned}$$

Unfortunately, because of the presence of the 1 and the 2 in (3.6), we find a large number of terms to evaluate (all of which may ultimately be reduced to simple functions of f_n):

$$\sum_{i_1 > i_2 > \dots > i_{n+2}=2}^{N-1} (i_1-i_2)(i_2-i_3) \dots (i_{n+1}-i_{n+2}) = f_{n+1}(N-2) \quad (3.15)$$

and

$$\begin{aligned} \sum_{i_1 > i_2 > \dots > i_{n+2}=2}^{N-1} (i_1-i_2)(i_2-i_3) \dots \\ \times (i_{n+1}-i_{n+2})(i_{n+2}-i_1) \\ = 2f_{n+2}(N-1)(2n+5)(N+n+1)^{-1} \\ + (1-N)f_{n+1}(N-2). \quad (3.16) \end{aligned}$$

With the use of (3.12), (3.13), (3.15), and (3.16), it is now possible to perform all the summations necessary within each order in (2.3). The summations between different orders becomes easy once we recognize the hypergeometric function emerging from the series:

$$\begin{aligned} \sum_{n=0}^{p-1} f_n(p) (-4)^n &= p {}_2F_1(p+1, 1-p; \frac{3}{2}; 1) \\ &= p(-1)^{p+1}, \\ \sum_{n=0}^{N-1} \sum_{p=n}^{N-1} f_{n-1}(p) (-4)^n &= {}_2F_1(N, 1-N; \frac{1}{2}; 1) \\ &= (2N-1)(-1)^{N+1}. \end{aligned}$$

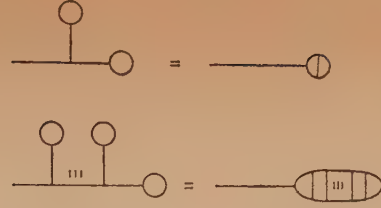


Fig. 2. Duality transformations which display the plausible divergence structure for multiloop amplitudes.

With these last two equations, it is now possible to perform all summations necessary to prove that (3.6) satisfies (2.4). The details themselves are quite tedious and will be omitted.

So far, we have only proven that $W_{P_1 P_2 \dots P_N}$ can reach 1 in the N -loop amplitude; only a slight alteration, however, allows us to find the other primitive set of divergences. For example, if we let $\bar{u}_i$ go to zero much faster than the other variables, then it is easy to show that

$$W_{P_1 P_2 \dots P_{i-1} P_{i+1} \dots P_N} = 1.$$

In other words, we can omit projective transformations in (2.4) at will by a simple change of limits. Since there are $2^N - 1$ ways in which projective operators may be omitted from (2.4), we find exactly that many divergences of the N -loop amplitude.

IV. CONCLUSION

In summary, we have proven that the $2^N - 1$ non-self-crossing paths mentioned earlier contribute divergences to the N -loop amplitude. The presence of these additional divergences could be expected both from the analog model and from the duality transformations exhibited in Fig. 2. (These figures seem to indicate that, depending on how one looks at a multiloop diagram, there are separate divergences associated with different clusters of loops.)

The absence of divergences other than the expected ones has not been demonstrated in the general case, because the geometric approach used in our previous paper cannot easily be extended to the N -loop case.

Because a detailed study of the $N=2$ case revealed no other divergence, it is likely that there are no additional divergences in the N -loop amplitude. If this last statement is true, then the renormalization program of Cremmer and Neveu seems likely to succeed.

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